Problem 1. For \( n \geq 0 \), give a combinatorial proof that \( f_{n+2} - 1 = f_0 + f_1 + \cdots + f_{n-1} + f_n \).

Problem 2. Let \( a_n \) denote the ways to color the cells of the \( n \times 1 \) board with the colors red, blue, and green such that no two adjacent cells may be colored red. Assuming \( a_0 = 1 \), find a recurrence relation for \( a_n \) and give justification that your relation is correct.

Problem 3. Find the number of \( n \)-length binary sequences \((x_1, x_2, x_3, \ldots, x_n)\) such that \( x_1 \leq x_2 \geq x_3 \leq \cdots \).

Problem 4. Let \( t_n \) denote the number of ternary strings, i.e., strings comprised of the numbers 0, 1, and 2, of length \( n \) which do not end with 00, 01, 11, or 10. Assuming \( t_0 = 1 \), find a recurrence relation for \( t_n \) and give justification that your relation is correct.

Bonus Problem 1. Let \( a_n \) denote the ways to color the cells of the \( n \times 1 \) board with the colors red, blue, and green such that the number of red cells is even and there must be at least one blue cell. Find a recurrence relation for \( a_n \) and give justification that your relation is correct.

Bonus Problem 2. Give a combinatorial proof of the following identity:

\[
f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i},
\]