In-Class Problems
Date: October 20 & 22

**Problem 1.** Let \( n \geq 0 \) and compute \( \sum_{i=0}^{n} \binom{n-i}{i} \) for \( n = 0, 1, 2, 3, 4 \). Conjecture a value for your sum and give a combinatorial proof that your conjecture is correct.

**Problem 2.** For \( n \geq 1 \), show that \( f_{n+1} \) counts the number of subsets of \([n]\) such that the subsets do not contain consecutive integers.

Note: There are two standard ways to generate a recurrence relation. The first is to find the first few terms of your sequence and then “guess” a recurrence. The second is to try to think about what is going on at the \( n^{th} \) step of the sequence and how that relates to previous steps. For example, when we considered the original tiling problem with \( T_n \), we began by finding \( T_1, T_2, T_3, \ldots \); however, we could have started with realizing that every tiling ends in either a square or domino. The second method is what you need to do when you want to justify that the relation you found is correct, so if you opt for the first way, then you eventually have to do the second step anyway.

**Problem 3.** For \( n \geq 0 \), show that \( f_n f_{n+1} = \sum_{k=0}^{n} f_k^2 \).
Problem 4. Without rewriting the terms using the recursion, give a combinatorial proof that \( 2f_n = f_{n+1} + f_{n-2} \).

Problem 5. Let \( a_n \) denote the ways to color the cells of the \( n \times 1 \) board with the colors red, blue, and green such that no two adjacent cells may be colored red. Assuming \( a_0 = 1 \), find a recurrence relation for \( a_n \) and give justification that your relation is correct.

Problem 6. For \( n \geq 1 \), suppose we are given \( 2n \) equally spaced points on a circle, and let \( p_n \) denote the number of ways to use line segments to connect these points in pairs such that the line segments do not intersect. Find a recurrence relation for \( p_n \) and give justification that your relation is correct.

Problem 7. For \( n \geq 0 \), show that \( f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} \).