Bonus Problems

Note: Bonus problems can be turned in for extra homework credit, but they must be worked on individually. You should not consult the Internet, other textbooks, etc. for help with these problems.

Bonus Problem 1. No longer available.

Bonus Problem 2. Give a combinatorial proof of the following identity:

\[ f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n}{i} \binom{n}{j}. \]

Bonus Problem 3. Give a combinatorial proof of the following identity:

\[ (n + 1)n2^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k}. \]

Bonus Problem 4. Given a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) of \([n]\), we say \( \sigma \) is 123-avoiding if there do not exist \( i < j < k \) such that \( \sigma_i < \sigma_j < \sigma_k \). Give a combinatorial proof to show that the number of 123-avoiding permutations of \([n]\) is \( C_n \), the \( n \)th Catalan number. As an example, 41532 is 123-avoiding but 41325 is not.

Bonus Problem 5. Joe thinks he’s watching too much t.v., so he decides to limit his usage. He decides that for the next 7 weeks, he will watch at least 1 hour of t.v. per day but no more than 11 hours in any calendar week, and he will only watch t.v. for a whole number of hours each day. Show that there is some consecutive string of days where Joe watches exactly 20 hours of television.

Bonus Problem 6. For any sequence of real numbers \( a_1, a_2, \ldots, a_{mnp+1} \), show that there either exists an increasing subsequence of length \( m + 1 \), a decreasing subsequence of length \( n + 1 \), or a constant subsequence of length \( p + 1 \), where \( m, n, p \in \mathbb{N} \).

Bonus Problem 7. No longer available.