MATH 3343-01 **Combinatorics** **Fall 2018**

Bonus Problems

Note: Bonus problems can be turned in for extra homework credit, but they must be worked on individually. You should not consult the Internet, other textbooks, etc. for help with these problems. All answers must be justified to receive **any** credit.

Bonus Problem 1. (8 points) Let $X = \{x_1, x_2, \ldots, x_{20}\}$. Determine the number of ordered triples (S_1, S_2, S_3) such that

$$S_1 \cup S_2 \cup S_3 = X \text{ and } S_1 \cap S_2 \cap S_3 = \emptyset.$$

Bonus Problem 2. (8 points) How many ways are there to arrange the elements of [2018] such that every number after the first is within one of a number somewhere to its left? For example, if 17 is the third number, then at least one of 16 or 18 must appear as the first or second number.

Bonus Problem 3. (8 points) For positive integers x and y, prove that $\dfrac{(x+y)!}{(x+y)^{x+y}} < \dfrac{x!y!}{x^x y^y}$.

Bonus Problem 4. (8 points) For $2 \le n \in \mathbb{N}$, give a combinatorial proof that

$$(n+1)n2^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k}.$$

Bonus Problem 5. (8 points) Let $n, m_1, m_2, m_3 \ge 0$. Find a formula for the following sum, defined as $S(n, m_1, m_2, m_3)$, and give a combinatorial proof that your formula is correct:

$$S(n, m_1, m_2, m_3) = \sum_{\substack{r,s,t \ge 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}.$$

Use your formula to explicitly compute $S(10, 5, 7, 10)$.

Bonus Problem 6. (8 points) Let a_n denote the ways to color the cells of the $n \times 1$ board with the colors red, blue, and green such that the number of red cells is even and there must be at least one blue cell. Find a recurrence relation for a_n and give justification that your relation is correct.

Bonus Problem 7. (8 points) Give a combinatorial proof of the following identity:

$$f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i}.$$

Bonus Problem 8. (8 points) Given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of $[n]$, we say σ is 123-*avoiding* if there do not exist $i < j < k$ such that $\sigma_i < \sigma_j < \sigma_k$. Give a combinatorial proof to show that the number of 123-avoiding permutations of $[n]$ is C_n, the n^{th} Catalan number. As an example, 41532 is 123-avoiding but $4\underline{1}3\underline{25}$ is not.

Bonus Problem 9. (8 points) Find the number of integer partitions of $n = 150$ that have exactly 60 parts.