Problem 1: Let \( S = \{1^\infty, 2^\infty, 3^\infty, 4^\infty\} \) and let \( a_n \) denote the number of \( n \)-permutations of \( S \) such that the number of 1’s is even, there is at least one 2, and the number of 4’s is odd.

i. By hand, list out the elements that contribute to \( a_1, a_2, \) and \( a_3 \).

ii. Compute a closed formula for \( a_n \) by computing \( A(x) \) egf \( \leftrightarrow \{a_n\}_{n \geq 0} \).

Problem 2: Let \( T = \{1^6, 2^5, 3^2, 4^3\} \) and let \( b_n \) denote the number of \( n \)-permutations of \( T \). Compute \( b_{10} \) using exponential generating functions, and determine how you would count this without egf’s.

Problem 3: Let \( A(x) \) egf \( \leftrightarrow \{a_n\}_{n \geq 0} \). Show that \( A'(x) \) egf \( \leftrightarrow \{a_{n+1}\}_{n \geq 0} \).

i. Show that \( A'(x) \) egf \( \leftrightarrow \{a_{n+1}\}_{n \geq 0} \).

ii. Consider the usual recurrence for the Fibonacci numbers, i.e., \( f_0 = f_1 = 1 \) and \( f_{n+2} = f_{n+1} + f_n \) for any \( n \geq 0 \). Use egf’s to find a closed formula for \( f_n \).

Problem 4: In 3343, we saw that if \( U(x) \) egf \( \leftrightarrow \{u_n\}_{n \geq 0} \), \( V(x) \) egf \( \leftrightarrow \{v_n\}_{n \geq 0} \), and \( W(x) \) egf \( \leftrightarrow \{w_n\}_{n \geq 0} \), with \( W(x) = U(x)V(x) \), then \( w_n = \sum_{k=0}^{n} u_k v_{n-k} \). Now suppose that \( A(x) \) egf \( \leftrightarrow \{a_n\}_{n \geq 0} \) and \( B(x) \) egf \( \leftrightarrow \{b_n\}_{n \geq 0} \).

i. If \( C(x) = A(x)B(x) \) and \( C(x) \) egf \( \leftrightarrow \{c_n\}_{n \geq 0} \), find \( \{c_n\} \).

ii. Using the identity \( e^{t(x+y)} = e^{tx} e^{ty} \) and part (i.), verify the Binomial Theorem by equating coefficients of \( t^n/n! \) in the egf’s of both functions.

iii. Give a combinatorial proof that \( n! = \sum_{k=0}^{n} \binom{n}{k} D_k \), where \( D_n \) represents the \( n \)-th derangement number, and then use this formula to find \( D(x) \), where \( D(x) \) egf \( \leftrightarrow \{D_n\}_{n \geq 0} \).