Problem 1: Let $\theta : S_n \rightarrow S_n$ be the bijection given in class such that $\text{inv}(\sigma) = \text{maj}(\theta(\sigma))$.

i. Find $\theta(491832756)$.

ii. Find $\theta^{-1}(698375241)$.

iii. In class we used the following claim to prove that $\theta$ was indeed the desired bijection which takes a permutation with $k$ inversions to a permutation with a major index of $k$:

If $\tau^{(i)} \in S_{n+1}$ is created by inserting $n+1$ into $\sigma \in S_n$ in the space labeled with $i$, then $\text{maj}(\tau^{(i)}) = \text{maj}(\sigma) + i$.

Provide a careful proof of this claim.

Problem 2: Define $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$ and let $[n]^*$ denote the set of all words using letters from $[n]$. Further, let $\phi : [n]^* \rightarrow [n]^*$ be the bijection given in class such that for any $w \in [n]^*$, $\text{maj}(w) = \text{inv}(\phi(w))$.

i. Find $\phi(1422314241)$.

ii. Find $\phi^{-1}(41213242)$.

Problem 3: Use Euler’s Pentagonal Number Theorem to find $p(16)$, the number of partitions of 16.