February Project #4, due by 5 p.m. on Monday, March 18^{th}

Reminder: This assignment must be done in LaTeX and emailed to me by the due date. Please send me the tex and pdf files, along with any files needed to compile your tex document, i.e., accompanying picture files. You are encouraged to use computer software for any of these projects.

****

Begin by reading the pdf document on formal power series (5 pages); moreover, recall that as a formal power series, we let

$$\frac{1}{1-\square} = \sum_{n\geq 0} \square^n.$$

Problem 1: Define

$$A_n(x,t) = \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{des}(\sigma)} t^{\mathrm{inv}(\sigma)} \text{ and } B_n(x,t) = \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{exc}(\sigma)} t^{\mathrm{maj}(\sigma)},$$

where $\mathcal{S}_n$ is the n-th symmetric group. Compute $A_n(x,t)$ and $B_n(x,t)$ for $n = 2, 3, 4$. Conjecture, but do not prove, a relationship between A and B.

****

The Ring of Symmetric Functions

For this entire section, suppose that we have N commuting variables, $x_1, x_2, \ldots, x_N$, and let $\mathbb{Q}[x_1, x_2, \ldots, x_N]$ denote the ring of polynomials in those N variables with coefficients in $\mathbb{Q}$. (*Note*: In this section, N will always denote the number of variables.)

Definition 1. A polynomial $p \in \mathbb{Q}[x_1, x_2, \ldots, x_N]$ is called *symmetic* if

$$p(x_1, x_2, \ldots, x_N) = p(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}),$$

for every $\sigma = \sigma_1\sigma_2\cdots\sigma_N \in \mathcal{S}_N$.

As an example, $p(x_1, x_2) = x_1 + x_1x_2 + x_2$ is a symmetric function in the two variables x_1 and x_2.

Definition 2. Let

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}^{N} (1 + x_i t).$$

Then $e_n = e_n(x_1, \ldots, x_N)$ is called the *n-th elementary symmetric function* in the variables $x_1, \ldots, x_N$.

Definition 3. Let

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i=1}^{N} \frac{1}{(1 - x_i t)}.$$

Then $h_n = h_n(x_1, \ldots, x_N)$ is called the *n-th homogeneous symmetric function* in the variables $x_1, \ldots, x_N$.

It is worth noting that these definitions make sense if the number of variables is infinite, but we wish to limit ourselves to the cases where the number of variables is finite.

Problem 2: Compute $e_4(x_1, x_2, x_3, x_4)$ and $h_2(x_1, x_2, x_3, x_4)$.

Problem 3: Find an equation which relates H and E. We want something of the form $H(t) =$ something with E, and my suggestion is to not overcomplicate this...it's just algebra on formal power series. Use this equation to prove (not combinatorially) that for any $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$$

Note: We would like to prove this combinatorially, but we don't have combinatorial interpretations for e_n and h_n.

****

Brick Tabloids

Definition 4. Let $n \in \mathbb{N}$. $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition* of n if each $\lambda_i \in \mathbb{N}$, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. This is denoted by $\lambda \vdash n$.

As an example, $\lambda = (1, 2, 2, 2, 5, 7, 7) = (1, 2^3, 5, 7^2) \vdash 26$. Given a partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we can picture λ as an array of left justified cells, where the i-th row from the top contains exactly λ_i cells. This is called the *Ferrers* diagram of λ. For example, the Ferrers diagram of the partition $\lambda = (3, 3, 4) \vdash 10$ is shown on the left side of Figure 1. A *column-strict tableaux of shape* λ is a Ferrers diagram of shape λ where the cells have been filled with integers such that the entries weakly increase within each row and strictly increase within each column, and we define the set of all column-strict tableaux of shape λ to be CS_λ. Given $T \in CS_\lambda$, we can define the weight of T, $w(T)$ as follows. For each cell, c, in T, if c contains the integer ℓ, then the cell c has a weight of $w(c) = x_\ell$. Then

$$w(T) := \prod_{c \in T} w(c).$$

For example, $T \in CS_{(3,3,4)}$ is shown on the right side of Figure 1, where $w(T) = x_1^3 x_2^3 x_3 x_4^2 x_6$.

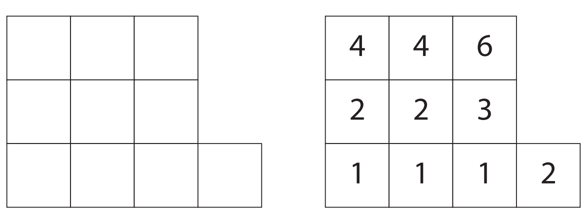

FIGURE 1. An example of the Ferrers diagram for the partition $\lambda = (3, 3, 4) \vdash 10$ and a corresponding column-strict tableaux of shape λ.

Problem 4: Draw all $T \in CS_{(1,2,4)}$ such that $w(T) = x_1^3 x_2^2 x_3 x_4$.

<u>Problem 5:</u> Consider the following partitions of n: $\lambda_n = (1^n) = (1, 1, \cdots, 1)$ and $\tau_n = (n)$. That is, the Ferrers diagram of λ_n is an array of n rows, each consisting of 1 cell, and the Ferrers diagram of τ_n is an array consisting of a single row of n cells. Compute

$$\sum_{T \in CS_{\lambda_4}} w(T) \text{ and } \sum_{T \in CS_{\tau_2}} w(T),$$

if we set $0 = x_5 = x_6 = \cdots$.

<u>Problem 6:</u> Compare the results of Problem 5 with those of Problem 2, and make a conjecture based on this comparison.