MATH 3343-01 **Combinatorics** **Spring 2020**

Homework 14

Due Date: April 14

You may use WolframAlpha or some other computational software to aid in the solving of any of these problems—and it is possible to check the correctness of many of your answers as well—but please include more than just an answer if you do that, i.e., give me some statement (i) that you used such a system and (ii) how you used it.

Reading: Niven: Chapter 6 and Section 7.1.

Problem 38. Fix $k \in \mathbb{N}$, and set $t_{n,k}$ to be the number of ways to select k integers from $[n]$ such that no two chosen integers are consecutive. Find a generating function for $\{t_{n,k}\}_{n\geq 1}$. Specifically state what a must be so that the coefficient of x^a is $t_{n,k}$.

Problem 39. On April 7 we showed that if $A(x) \longleftrightarrow \{a_n\}$ and $B(x) \longleftrightarrow \{b_n\}$, then $A(x)B(x) \longleftrightarrow \{c_n\}$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Use this fact to find an ogf for the Catalan numbers.
(Note: If done correctly, you will get two possible answers for this ogf, but only one can be correct. Be specific about how to mathematically choose the right one.)

Problem 40. A *standard tetrahedral die* is a regular tetrahedron that has the numbers 1, 2, 3, 4 on its sides, and the number that is "rolled" is the number that is on the bottom face. A person wishes to make two nonstandard tetrahedral die—ones that do not have all of 1, 2, 3, and 4 on their faces, and possibly different from each other—such that when this pair of dice is rolled, the probability of the sum of the rolled faces being S for each $2 \leq S \leq 8$ is the same as if two standard tetrahedral dice were rolled. Use generating functions to show whether or not this can be done.

Problem 41. Let $A(x) \longleftrightarrow \{a_n\}$.

i. If $S(x) = \dfrac{A(x)}{1-x}$, for what sequence is $S(x)$ a generating function.

ii. Let $H_1, H_2, H_3, \ldots$ denote the sequence of *Harmonic numbers*, defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}, \text{ for } n \geq 1.$$

Find a generating function for $\{H_n\}$.
(Hint: A result from Assignment #13 will be helpful here.)

Problem 42. Given $n \in \mathbb{N}$, we say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an *integer partition of n into k parts* if each $\lambda_i \in \mathbb{N}$ with

i. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, and

ii. $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The λ_i's are called the *parts* of the partition. For example, $\lambda = (5, 2, 2)$ is an integer partition of 9 into 3 parts: $\lambda_1 = 5$ and $\lambda_2 = \lambda_3 = 2$. If p_n is the number of integer partitions of n, find p_n for $1 \leq n \leq 6$.
(Note: You may want to just compute these numbers by hand, as p_6 is not that large.)