Definition: Let $A, B$ be sets. We say that $B$ is a subset of $A$, denoted by $B \subseteq A$, if whenever $x \in B$, we also have that $x \in A$. We say that $A = B$ if $A$ and $B$ are subsets of each other.

Definition: The empty set, denoted by $\emptyset$, is the set which contains no elements.

Definition: Let $A, B, C$ be sets in some universal set $X$.

i. The union of $A$ and $B$ is $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$.

ii. The intersection of $A$ and $B$ is $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$.

iii. The complement of $A$ in $B$ is $B - A = \{ x \mid x \in B \text{ and } x \notin A \}$.

iv. The complement of is $A^c = \{ x \mid x \notin A \} = X - A$.

We will now give some important facts about sets.

Proposition 1. Let $A, B, C$ be sets.

i. $A \subseteq A$.

ii. $\emptyset \subseteq A$.

iii. If $C \subseteq B$ and $B \subseteq A$, then $C \subseteq A$.

iv. $A, B \subseteq A \cup B$.

v. $A \cap B \subseteq A, B$.

vi. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

vii. $A \cap (B - A) = \emptyset$.

viii. (associative) $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$.

ix. (distributive) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Proof of the first distributive law. \((\subseteq)\) Pick \(x \in A \cup (B \cap C)\). Then \(x \in A\) or \(x \in (B \cap C)\). If \(x \in A\), then \(x \in A \cup B\) and \(x \in A \cup C\), giving that \(x \in (A \cup B) \cap (A \cup C)\). If \(x \notin A\), then it must be the case that \(x \in B \cap C\), giving that \(x \in B\) and \(x \in C\), and so \(x \in A \cup B\) and \(x \in A \cup C\). Thus, \(x \in (A \cup B) \cap (A \cup C)\), or \(A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)\).

\((\supseteq)\) Pick \(x \in (A \cup B) \cap (A \cup C)\). Then \(x \in A \cup B\) and \(x \in A \cup C\). If \(x \in A\), then \(x \in A \cup (B \cap C)\). Otherwise, \(x \notin A\), and then it must be the case that \(x \in B\) and \(x \in C\), that is, \(x \in B \cap C\). Thus, \(x \in A \cup (B \cap C)\), and \((A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)\).

Therefore, \((A \cup B) \cap (A \cup C) = A \cup (B \cap C)\). \(\Box\)

**Definition:** Let \(A\) be a set. The **power set** of \(A\) is \(\mathcal{P}(A) = \{B \mid B \subseteq A\}\).

**Proposition 2.** If \(A\) is a set containing \(n \geq 0\) elements, then the power set of \(A\), \(\mathcal{P}(A)\) contains \(2^n\) elements.

**Definition:** Let \(A, B\) be sets. The **direct product** of \(A\) and \(B\) is

\[ A \times B = \{(a, b) \mid a \in A\ \text{and}\ b \in B\}. \]