Exam I Review Problems

**Review Problem 0:** Give definitions for the following terms: statement, set, subset, union, intersection, complement, direct product, relation, equivalence relation, equivalence class, partition. Study all of the proofs given in class on the homework.

**Review Problem 1:** State the Principle of Mathematical Induction. State the Well-Ordering Principle.

**Review Problem 2:** For statements $P$, $Q$, and $R$, determine the truth table for $(P \lor Q) \lor (Q \rightarrow \neg R)$.

**Review Problem 3:** For each of the following arguments, assume all statements preceding the final one are true. Determine the validity of the final statement.

i. Either John is busy or he is sick. If he is busy, then he is exhausted. He is not exhausted. Therefore he is sick.

ii. If the Royals win, then Kansas City will celebrate, and if the Orioles win, Baltimore will celebrate. Either the Royals will win or the Orioles will win. However, if the Royals win, then Baltimore will not celebrate, and if the Orioles win, Kansas City will not celebrate. So, Baltimore will celebrate if and only if Kansas City does not celebrate.

**Review Problem 4:** Consider the following statement, also known as Fermat’s Last Theorem:

Given $n \in \mathbb{N}$ with $n > 2$, $x^n + y^n = z^n$ has no solutions for any nonzero $x, y, z \in \mathbb{Z}$.

i. Rewrite this statement as an implication.

ii. Write the converse and contrapositive of the statement.

iii. Write the negation of the statement.

**Review Problem 5:** Let $f : X \to \mathbb{R}$ be a function. We say that $f$ is uniformly continuous on $A \subseteq X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in A$ satisfying $|x - y| < \delta$ we have that $|f(x) - f(a)| < \epsilon$. In a meaningful way say what it means for $f$ to not be uniformly continuous on $A$.

**Review Problem 6:** Let $n \in \mathbb{N}$. Conjecture and prove a closed form expression for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$
**Review Problem 7:** State and prove DeMorgan’s Laws for sets.

**Review Problem 8:** Let $A$ and $B$ be sets. State what it means for $x$ to not be an element of $A - B$. State what it means for $x$ to not be an element of $A \cap B$. State what it means for $x$ to not be an element of $A \times B$.

**Review Problem 9:** List all of the possible equivalence relations on the set $A = \{1, 2, 3, 4\}$.

**Review Problem 10:** Let $L = \{(a, b) \mid ab > 0\} \subseteq \mathbb{Z}^2$. Determine whether or not $L$ is an equivalence relation on $\mathbb{Z}$, and if it is, then describe, in words, $[-17]$.

**Review Problem 11:** Given $f : \mathbb{R} \to \mathbb{R}$, we can define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $f(x) = f(y)$. (Recall that functions will NOT be on the exam.) Determine whether or not $\sim$ is an equivalence relation on $\mathbb{R}$, and if it is, then describe, in words, $[a]$.

**Review Problem 12:** [Characterization of Union] Given sets $A$ and $B$, let $X$ be a set with the following properties:

i. $A \subseteq X$ and $B \subseteq X$.

ii. For any set $Y$, if $A \subseteq Y$ and $B \subseteq Y$, then $X \subseteq Y$.

Show that $X = A \cup B$. Can you formulate a similar characterization for $A \cap B$?

**Review Problem 13:** On each planet in a planetary system consisting of an odd number of planets there is a single astronomer observing the nearest planet. The distances between each pair of planets are all different. Prove that at least one planet is not observed by any astronomer.

**Review Problem 14:** Given $0 \leq x \leq \pi$, show that for all $n \in \mathbb{N}$, $|\sin(nx)| \leq n \sin(x)$. You may use any of the three facts below to aid in proving this statement.

i. For $a \in \mathbb{R}$, $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

ii. *(The Triangle Inequality.)* For $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

iii. For $a, b \in \mathbb{R}$, $\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$. 