Third Exam Homework Assignments

**Problem 1.** Show that the set $\mathbb{Q} \times \mathbb{R}$ is infinite.

**Problem 2.** For $a, b, \in \mathbb{R}$ with $a < b$, set $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$. Show that $[0, 1]$ is infinite.

**Problem 3.** Define $\phi : \mathbb{N} \to \mathbb{Z}$ by

$$
\phi(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \text{ is even, and} \\
\frac{1-x}{2} & \text{if } x \text{ is odd.}
\end{cases}
$$

Prove that $\phi$ is a bijection.

**Problem 4.** Show that the set of positive, even integers is countable.

**Problem 5.** Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$$
f(x) = \begin{cases} 
(i, j) & \text{if } x = 2^i 3^j, \text{ and} \\
(1, 1) & \text{otherwise.}
\end{cases}
$$

Prove that $f$ is a surjection, showing that $\mathbb{N} \times \mathbb{N}$ is countable.

**Problem 6.** Show that $\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k-times}$ is countable.

**Problem 7.** Suppose that $A, B$ are countable sets. Show that $A \cup B$ is countable. (Hint: It may be easier to use the fact that $\mathbb{N} \times \mathbb{N}$ is countable as opposed to using $\mathbb{N}$.)

**Problem 8.** Given $a, b \in \mathbb{R}$ with $a < b$, show that $(a, b) \sim (0, 1)$. 
Problem 9. Suppose that $A$ is an infinite countable set and pick $x \in A$. Show that $A \sim A - \{x\}$.

Problem 10. Suppose that $A \subseteq B$ with $A$ uncountable. Show that $B$ is uncountable.

Problem 11. (a) Find two binary expansions of the real number $x = 0.34375$.

(b) Show that in a binary expansions $0.01$ and $0.00111$ represent the same real number.

(c) Using the fact that $\frac{x}{1-x} = \sum_{i=1}^{\infty} x^i$ for every $|x| < 1$, show that the binary representation of $\frac{1}{7}$ is $0.001101$. (Hint: $7 = 2^3 - 1$.)

Problem 12. (a) Use Cantor’s diagonalization method to show that the set of infinite sequences of 0’s and 1’s is uncountable. That is, $\mathcal{B} = \{x_1x_2x_3\ldots \mid x_i = 0 \text{ or } 1 \forall i \in \mathbb{N}\}$ is uncountable.

(b) Notice that if we remove the sequences 000\ldots and 111\ldots from $\mathcal{B}$ then each of the remaining sequences corresponds to a binary decimal representation of some $x \in (0, 1)$. For example, the sequence 001001 \in $\mathcal{B}$ corresponds to the binary decimal $0.001001 = \frac{1}{7}$. So, does your proof in part (a) show that $(0, 1)$ is uncountable? Explain your response.

Problem 13. Consider the characterization of supremum stated in class. State a similar theorem characterizing the infimum of a set.

Problem 14. For each of the following sets, determine, without justification, the sup and inf of each set, if they exist. For each problem, say whether or not numbers you found where in the set or not.

(a) $\mathbb{N}$
(b) $\mathbb{Q}^+$
(c) $\mathbb{R}^+$
(d) $\mathbb{Q}^*$
(e) $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$
(f) $\left\{ \frac{n^2}{n^2+1} \mid n \in \mathbb{N} \right\}$
(g) $\left\{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \right\}$
**Problem 15.** On the last assignment you found a characterization of *infimum*. Prove that theorem.

**Problem 16.** Determine, with justification, each of the following:

(a) $\inf \mathbb{N}$

(b) $\sup \{2 - \frac{1}{n} \mid n \in \mathbb{N}\}$

(c) $\inf \mathbb{R}^*$

**Problem 17.** Prove the following corollary to the Archimedean Property: *If $a$ is a positive real number, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$.***