Exam II Review Problems

Review Problem 1: Give definitions for the following terms: function, injective, surjective, bijective, binary operation, group, Abelian, identity, inverse, cyclic group. State the Division Algorithm. State the Well-Ordering Principle.

Review Problem 2: Given a function \( f : X \to Y \), for \( Z \subseteq X \) we define the set \( f(Z) = \{ f(z) | z \in Z \} \subseteq Y \). Consider \( A, B \subseteq X \).

a. Show that \( f(A \cup B) = f(A) \cup f(B) \).

b. Show that \( f(A \cap B) \subseteq f(A) \cap f(B) \).

c. Find an example where \( f(A \cap B) \neq f(A) \cap f(B) \).

Review Problem 3: Let \( X \) and \( Y \) be sets. Prove the following statements.

a. The function \( f : X \to Y \) is an injection if and only if there exists a function \( g : Y \to X \) so that \( g \circ f = Id_X \). \( g \) is called a left inverse.

b. The function \( f : X \to Y \) is a surjection if and only if there exists a function \( g : Y \to X \) so that \( f \circ g = Id_Y \). \( g \) is called a right inverse.

Review Problem 4: Consider the function \( h : X \to Y \) defined by \( h(x) = x^2 - 2x \).

a. Show that if \( X = Y = \mathbb{R} \), then \( h \) is neither injective nor surjective.

b. Find (with proof) \( X, Y \subseteq \mathbb{R} \) such that \( h \) is surjective but not injective. Can you find the “largest” \( Y \)?

c. Find (with proof) \( X, Y \subseteq \mathbb{R} \) so that \( h \) is bijective. Can you find the “largest” \( X, Y \)?

Review Problem 5: Let \( n \in \mathbb{N} \) and suppose \( X \) is a set with \( n \) elements. Find, with proof, the number of bijections from \( X \) to itself.

Review Problem 6: Suppose \( X, Y \) are sets and let \( B(X, Y) \) denote the set of all bijections from \( X \) to \( Y \). Show that \( (B(\mathbb{R}, \mathbb{R}), \circ) \) is a group. Determine whether or not \( (B(\mathbb{Z}, \mathbb{Z}), \circ) \) is a group. What about \( (B(\mathbb{R}, \mathbb{Z}), \circ) \)?

Review Problem 7: Fix \( a, b \in \mathbb{N} \). Show that \( H_{a,b} = \{ am + bn \mid m, n \in \mathbb{Z} \} \) is a subgroup of \((\mathbb{Z}, +)\).
Review Problem 8: Let $G$ be a group and suppose that $a \in G$ such that $|a| = n$. Show that $a^i = a^j$ if and only if $n|(i - j)$.

Review Problem 9: Suppose $G$ is a group and $a, b \in G$. Show that if $|ab| = n$, then $|ba| = n$.

Review Problem 10: Is $U(18)$ cyclic? If so, find all of its generators.

Review Problem 11: Suppose $n > 1$. What is the order of the element $2^n - 1$ in the group $U(2^n)$?

Review Problem 12: For groups $(G, \ast_G), (H, \ast_H)$, define $G \oplus H = \{(g, h) \mid g \in G \text{ and } h \in H\}$. Show that $(G \oplus H, \cdot)$ is a group if we define $(g, h) \cdot (g', h') = (g \ast_G g', h \ast_H h')$.

Review Problem 13: Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is cyclic but $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not.


Review Problem 15: Determine all subgroups of the group $G = (\mathbb{Z}, +)$.

Review Problem 16: Let $G$ be a group and suppose $H \leq G$. Define

$$C(H) = \{x \in G \mid xh = hx \forall h \in H\}.$$ 

Show that $C(H) \leq G$.

Review Problem 17: Suppose that $G, H, K$ are groups such that $G \approx H$ and $H \approx K$. Prove that $G \approx K$. 