Exam I Review Problems

**Review Problem 1:** Give definitions for the following terms: *statement, set, subset, union, intersection, complement, direct product, relation, equivalence relation, equivalence class, partition.*

**Review Problem 2:** State the Principle of Mathematical Induction.

**Review Problem 3:** For statements $P$, $Q$, and $R$, determine the truth table for $(P \lor Q) \lor (Q \rightarrow \neg R)$.

**Review Problem 4:** For each of the following arguments, assume all statements preceding the final one are true. Determine the validity of the final statement.

a. Either John is busy or he is sick. If he is busy, then he is exhausted. He is not exhausted. Therefore he is sick.

b. If the Dodgers win, then Los Angeles will celebrate, and if the White Sox win, Chicago will celebrate. Either the Dodgers will win or the White Sox will win. However, if the Dodgers win, then Chicago will not celebrate, and if the White Sox win, Los Angeles will not celebrate. So, Chicago will celebrate if and only if Los Angeles does not celebrate.

**Review Problem 5:** Consider the following statement, also known as Fermat’s Last Theorem: *Given $n \in \mathbb{N}$ with $n > 2$, $x^n + y^n = z^n$ has no solutions for any nonzero $x, y, z \in \mathbb{Z}$.*

i. Rewrite this statement as an implication.

ii. Write the converse and contrapositive of the statement.

iii. Write the negation of the statement.

**Review Problem 6:** Study all of the proofs given in class on the homework.

**Review Problem 7:** Let $n \in \mathbb{N}$. Conjecture and prove a closed form expression for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

**Review Problem 8:** Prove DeMorgan’s Laws for sets.
Review Problem 9: Let $A$ and $B$ be sets. State what it means for $x$ to not be an element of $A - B$. State what it means for $x$ to not be an element of $A \cap B$. State what it means for $x$ to not be an element of $A \times B$.

Review Problem 10: List all of the equivalence relations on the set $A = \{1, 2, 3, 4\}$.

Review Problem 11: Let $L = \{(a, b) \mid ab > 0\} \subseteq \mathbb{Z}^2$. Determine whether or not $L$ is an equivalence relation on $\mathbb{Z}$, and if it is, then describe, in words, $[-17]$.

Review Problem 12: Given $f: \mathbb{R} \to \mathbb{R}$, we can define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $f(x) = f(y)$. (Recall that functions will NOT be on the exam.) Determine whether or not $\sim$ is an equivalence relation on $\mathbb{R}$, and if it is, then describe, in words, $[a]$.

Review Problem 13: [Characterization of Union] Given sets $A$ and $B$, let $X$ be a set with the following properties:

i. $A \subseteq X$ and $B \subseteq X$.

ii. For any set $Y$, if $A \subseteq Y$ and $B \subseteq Y$, then $X \subseteq Y$.

Show that $X = A \cup B$. Can you formulate a similar characterization for $A \cap B$?

Review Problem 14: On each planet in a planetary system consisting of an odd number of planets there is a single astronomer observing the nearest planet. The distances between each pair of planets are all different. Prove that at least one planet is not observed by any astronomer.

Review Problem 15: Given $0 \leq x \leq \pi$, show that for all $n \in \mathbb{N}$, $|\sin(nx)| \leq n\sin(x)$. You may use any of the facts below to aid in proving this fact.

1. For $a \in \mathbb{R}$, $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

2. (The Triangle Inequality) For $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

3. For $a, b \in \mathbb{R}$, $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.