Homework 11

Due in class on April 21

We define that a set X is *finite* if $X = \emptyset$ or for some $n \in \mathbb{N}$, there exists a bijection f between X and $\mathbb{I}_n := \{1, 2, \ldots, n\}$. We then define that X is *infinite* if X is not finite. We further define that X is *countable* if X is finite or if there exists a bijection φ between X and $\mathbb{N}$, and we define that X is *uncountable* if X is not countable. Finally, we say that the *cardinality of* X, denoted $\text{card}(X)$, is some "measure" of the size of X; in particular, for a finite set X, we set $\text{card}(X) = |X|$. We say that given two sets A and B, $\text{card}(A) = \text{card}(B)$ if there exists a bijection between A and B. With this definition, we see that it is possible that two infinite sets may not have the same cardinality.

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Problem 1. Prove that if A is a set such that $|A| = n \in \mathbb{N}$, then the number of bijections from A to itself is $n!$. Recall that $0! = 1$, and for $n \in \mathbb{N}$, $n! = n(n-1)(n-2)\cdots(2)(1)$.

Problem 2. Below is a list of 10 ten-digit numbers made up of only 2's and 4's. Some digits are visible to you, while others are not. Find a ten-digit number made of only 2's and 4's that is not on the list, and give a **clear** explanation as to why your answer is not on the list.

$$\begin{array}{llllllllll}
n_1 = x & x & x & 2 & x & 4 & x & x & x & 4 \\
n_2 = x & x & x & x & 4 & x & 2 & x & 4 & x \\
n_3 = x & x & x & x & x & x & x & 4 & x & x \\
n_4 = 4 & x & 2 & x & x & x & 2 & x & x & x \\
n_5 = x & 2 & x & x & x & 2 & x & x & x & x \\
n_6 = x & x & 4 & x & 2 & x & x & x & x & x \\
n_7 = x & x & x & 4 & x & x & x & x & x & x \\
n_8 = x & x & 2 & x & x & x & x & x & x & x \\
n_9 = x & 2 & x & x & 4 & x & 2 & x & x & x \\
n_{10} = 4 & x & x & x & x & 4 & x & x & x & x
\end{array}$$

Problem 3. Suppose $f : X \to Y$ is an injection and X is infinite. Prove that Y is infinite.

Problem 4. Consider the relation $\sim = \{(A, B) \mid \text{card}(A) = \text{card}(B)\} \subseteq \mathcal{P}(\mathbb{R})^2$, where $\mathcal{P}(\mathbb{R})$ is the power set of the real numbers. Prove or disprove that $\sim$ is an equivalence relation.

Problem 5. For this problem $(u, v) = \{x \in \mathbb{R} \mid u < x < v\}$, that is, (u, v) is the continuous, real interval from u to v, including neither u nor v.

i. Show that for any $a, b \in \mathbb{R}$ with $a < b$, $\text{card}((0, 1)) = \text{card}((a, b))$.

ii. Prove that $f : (-1, 1) \to \mathbb{R}$ by $x \mapsto \frac{x}{1-x^2}$ is a bijection.
(Note that these two problems show that $\text{card}((0, 1)) = \text{card}(\mathbb{R})$.)

Problem 6. Let X be an infinite set. Prove that the following three statements are equivalent.

i. X is a countable set.

ii. There exists a surjection $f : \mathbb{N} \to X$.

iii. There exists an injection $g : X \to \mathbb{N}$.