In-Class Problem Day #1 - With Selected Solutions
Date: February 1

Problem 1. The ISBN-10 code on the back of most books is a 10-digit number written as

\[ a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}, \]

where \( a_{10} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X\} \) (the \( X \) represents a 10) is known as a check digit. The check digit is determined by the first nine digits using the formula

\[ a_{10} \equiv \sum_{i=1}^{9} i a_i \pmod{11}. \]

If the first nine digits of the ISBN-10 of your textbook is 0-486-46931, then without looking at your textbook, find the check digit.

Problem 2.

i. List all primes numbers that divide 50!.

ii. Find the remainder obtained by dividing the number \( S = 1! + 2! + 3! + \cdots + 50! \) by 9.

iii. Find the remainder obtained by dividing the number \( T = 1^5 + 2^5 + 3^5 + \cdots + 100^5 \) by 4.

Solution to iii: Notice that if \( n \) is even, then there exists \( k \in \mathbb{Z} \) such that \( n = 2k \), giving us that \( n^5 = (2k)^5 = 32k^5 \equiv 0 \pmod{4} \). Thus,

\[
T = 1^5 + 2^5 + 3^5 + \cdots + 99^5 + 100^5 \equiv 1^5 + 0 + 3^5 + \cdots + 99^5 + 0 \pmod{4} \\
\equiv 1^5 + 3^5 + \cdots + 97^5 + 99^5 \pmod{4} \\
\equiv 1^5 + (-1)^5 + \cdots + 1^5 + (-1)^5 \pmod{4} \\
\equiv 0 \pmod{4}.
\]
**Problem 3.** Suppose the complete list of primes is \( p_1, p_2, \ldots, p_k \). Prove each of the following, which would give alternate proofs of the infinitude of primes.

i. Let \( M = p_k! - 1 \), and show none of the \( p_i \) divide \( M \).

ii. Let \( N = p_2p_3 \cdots p_k + p_1p_3 \cdots p_k + \cdots + p_1p_2 \cdots p_{k-1} \), and show none of the \( p_i \) divide \( N \).

**Solution to ii:** For \( 1 \leq a \leq k \), define \( P_a := \frac{1}{p_a} \sum_{i=1}^{k} p_i \), so that \( N = P_1 + P_2 + \cdots + P_k \). Pick some prime divisor of \( N \), say \( p_j \). By definition, \( p_j \) divides \( P_a \) for every \( a \neq j \). By a previous theorem, \( p_j \) then divides \( N - (P_1 + P_2 + \cdots + P_{j-1} + P_{j+1} + \cdots + P_k) = P_j \), that is, \( P_j = p_jx \) for some \( x \in \mathbb{Z} \). But \( P_j = p_1p_2 \cdots p_{j-1}p_{j+1} \cdots p_k \), which does not have \( p_j \) as a prime factor, a contradiction.

**Problem 4.** A man with no money in his pocket cashes a check at a bank, and the teller mistakes the number of cents for the number of dollars, and vice versa. Unaware, the man then goes out and buys a candy bar for 68 cents. After this purchase, he realizes he has twice the amount of the original check. Determine the smallest amount for which the check could have been written.

**Solution:** Let \( d \) denote the number of dollars and \( c \) the number of cents for the check. Then the check was written for an amount of \( 100d + c \) cents. By switching the dollars and cents, the teller gave the man \( 100c + d \) cents, and we know from the problem that

\[
100c + d - 68 = 2(100d + c) \Rightarrow 98c - 199d = 68.
\]

We are looking for solutions to this equation where \( c, d > 0 \). Using the Euclidean Algorithm, we see that \((98, 199) = 1\), and back substituting gives us that \( 98(-67) + (199)(-33) = 1 \). Multiplying both sides by 68 gives \( 98(-4556) + (199)(-2244) = 68 \), and so every solution is of the form \( c = -4556 - 199t, d = -2244 - 98t \). Setting \( \hat{c}, \hat{d} > 0 \) gives that \( -23 \geq t \in \mathbb{Z} \). Plugging in \( t = -23 \), we get that \( c = 21, d = 10 \), so the original check was for $10.21.