Problem 1: Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for every real number $a$. Here, $[v]$ denotes the greatest integer less than or equal to $v$.

Problem 2: Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. For example, $23 = 9 + 8 + 6 = 3^2 + 2^3 + 2 \times 3$.

Problem 3: Basketball star Shanille O’Keal’s team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80%, but by the end of the season, $S(N)$ was more than 80%. Was there necessarily a moment in between when $S(N)$ was exactly 80%?

Problem 4: Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\text{det} \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that $u_n$ is an integer for all $n$. (By convention, $0! = 1$.)

Problem 5: Show that for each positive integer $n$,

$$n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\},$$

where lcm denotes the least common multiple and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

Problem 6: Find all positive integers $n$, $k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$