Real Analysis Exchange

Eddy Kwessi, Department of Mathematics, Trinity University, San Antonio, Texas 78212, U.S.A. email: ekwessi@trinity.edu

ATOMIC DECOMPOSITION OF LORENTZ-BOCHNER SPACES AND APPLICATIONS

Abstract

In this paper, we show that under certain conditions, the Lorentz-Bochner spaces have an atomic decomposition. This atomic decomposition is then used for the study of the boundedness of some operators on these spaces such as the Hardy-Littlewood maximal operator, the Hilbert transform, the multiplication and composition operators. A version of Marcienkiewicz interpolation Theorem for Lorentz-Boncher spaces is also revisited.

1 Introduction.

In his papers [12] and [13], G. G. Lorentz introduced the now famous Lorentz Spaces $L_{p,q}$ in the early 1950s, as a generalization of the L_p spaces. It is known that in general these spaces are quasi-Banach spaces. The Lorentz-Bochner spaces are some variants of the Lorentz spaces defined on σ-finite measure spaces with Banach spaces-valued functions. Atomic decompositions of Banach spaces have been studied before by many authors for various purposes. For instance in [5], R. R. Coifman proved that the Hardy's space $H^1(\mathbb{D})$ has an atomic decomposition, a result that can be used to give another proof of the famous theorem by C. Fefferman [8] that $(H^1(\mathbb{D}))^* = BMO$. Weisz [17] studied atomic decompositions of martingales Hardy's spaces, Liu and Hou [14] studied the atomic decomposition of Banach-vector-valued martingales spaces, Yong, Lihua and Peide [18] studied atomic decompositions of Lorentz martingale spaces.

Mathematical Reviews subject classification: Primary: 42A55, 42A45; Secondary: 46E30
Key words: atomic, operators, Lorentz-Bochner, Multiplication, Composition, interpolation

In this paper, we present an atomic decomposition for the Lorentz-Bochner space, defined on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ with X-valued functions, where X is a Banach space. This atomic decomposition can be an effective tool to prove the boundedness of operators such as the the Hardy-Littlewood operator, the Hilbert transform, the multiplication, the composition operators acting on theses spaces. Indeed, the boundedness of these operators is reduced to the boundedness on characteristic functions. We will also give another proof of a version of Marcienkiewicz Interpolation Theorem on the Lorentz-Bochner spaces.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space with

$$\Omega = \cup_{n=1}^{\infty} \Omega_n, \quad \text{where } \Omega_n \cap \Omega_m = \emptyset, \quad n \neq m. \tag{1}$$

Definition 1. Let X be a Banach space. We define for a measurable function $f : \Omega \to X$ the *decreasing rearrangement* of f as the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{y > 0 : d(f, y) \leq t\} ,$$

where $d(f, y) = \mu(\{x : |f(x)| > y\})$ is the *distribution* of the function f.
For $t > 0$, let

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad f^{**}(0) = f^*(0).$$

We also define on Ω the function $\|f\|$ by $\|f\|(\omega) = \|f(\omega)\|,\ \omega \in \Omega$.

Definition 2. Given a strongly measurable function f, define

$$\|f\|_{p,q} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^{\infty} \left(t^{\frac{1}{p}} ||f||^{**}(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 1 < p < \infty, \quad 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} ||f||^{**}(t) & \text{if } 1 < p \leq \infty, \quad q = \infty. \end{cases}$$

The set of all functions f with $\|f\|_{p,q} < \infty$ is called *the Lorentz-Bochner Space* with indices p and q and denoted by $L_{p,q}(\Omega, X)$. We know that endowed with this norm, the Lorentz-Bochner spaces are Banach spaces. Recall that for $1 < p, q < \infty$, we have $L^*_{p,q}(\Omega, X) = L_{p',q'}(\Omega, X^*)$ and $L^*_{p,1}(\Omega, X) = L_{p,\infty}(\Omega, X^*)$ where p' and q' are the Hölder conjugates of p and q respectively and X^* is the dual space of X. In the sequel, we will refer to $L_{p,q}(\Omega, X)$ as $L_{p,q}$ for simplicity.

Remark 3. We will make the assumption throughout this paper that μ is a separable measure, that is, there is a countable family $\mathcal{M}$ of sets from $\mathcal{A}$ of finite measure such that for any $\epsilon > 0$ and any set $A \in \mathcal{A}$ of finite measure, we can find $B \in \mathcal{M}$ with $\mu(A \Delta B) < \epsilon$.

Remark 4. We can choose the Ω_n's in (1) such that

$$\|\chi_{\Omega_n}\|_{p,q} = 1,\ \forall n \in \mathbb{N},\ 1 < p < \infty, 1 < q \leq \infty. \tag{2}$$

where χ_{Ω_n} is the characteristic function on the μ-measurable set Ω_n. In fact, for all μ-measurable $A \subset \Omega$, we have

$$\|\chi_A\|_{p,q} = \mu(A)^{1/p}.$$

Thus, by normalization we can choose Ω_n such that (2) is satisfied.

Definition 5. Recall that the centered Hardy-Littlewood maximal operator on $L_{p,q}$ is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \|f(t)\| d\mu(t),$$

where $B_r(x)$ is the ball of X centered at x with radius r.
For a locally integrable function f on $\mathbb{R}$, define the Hilbert transform H on $L_{p,q}(\mathbb{R}, \mathbb{R})$ as

$$Hf(x) = \frac{1}{\pi} \text{p.v} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt, \quad \text{for all } x \in \mathbb{R}.$$

For a strongly bounded measurable function $g : \Omega \to \mathcal{B}(X)$ (the set of bounded functions on X), we define the multiplication operator $M_g : L_{p,q} \to L_{p,q}$ as

$$M_g f(\omega) = g(\omega)(f(\omega)), \quad \text{for } \omega \in \Omega, \text{ and for } f \in L_{p,q}.$$

For a non-singular measurable transformation $h : \Omega \to \Omega$, we define the composition operator on $L_{p,q}$ as

$$C_h f(\omega) = (f \circ h)(\omega), \quad \text{for } \omega \in \Omega, \text{ and for } f \in L_{p,q}.$$

A linear or quasi-linear operator T is said to of type (p, q) if $T : L_{p,q} \to L_{p,q}$ is bounded. T will be of restricted-type (p, q) if $T : L_{p,1} \to L_{p,q}$ is bounded.
For two Banach spaces X_1 and X_2, we will denote $\|x\|_{X_1} \cong \|x\|_{X_2}$ to say that there are two absolute constants C_1, C_2 such that

$$C_1\|x\|_{X_1} \leq \|x\|_{X_2} \leq C_2\|x\|_{X_1}.$$

We will denote through out this paper $\mathbb{N}$ as the set of positive integers and $l^1(X)$ as the set of sequences $\{x_n \in X\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$.

2 Main Results.

Our first result, which is a generalization of Theorem 2.1 in [9] provides a candidate for the atomic decomposition of Banach spaces. The proof is in the same line to that of Theorem 2.1 in [9] up to some minor adjustments and will be omitted for sake of brevity.

Theorem 6. *Let K be a real or complex Banach space, and $\mathcal{B} = (b_i)_{i\in I}$ a (not necessarily countable) bounded family in K. Let $\Phi : K \to K$ be a bounded function. Define*

$$\mathcal{K}_{\mathcal{B},\Phi} = \left\{ f = \sum_{i\in I} x_i \Phi(b_i) : \quad x_i \in \mathbb{R} \text{ or } \mathbb{C}, \ \sum_{i\in I} |x_i| < \infty \right\}.$$

Let a norm on $\mathcal{K}_{\mathcal{B},\Phi}$ be defined as

$$\|f\|_{\mathcal{K}_{\mathcal{B},\Phi}} = \inf \sum_{i\in I} |x_i|, \ \text{ where the infimum is taken over all representation of } f.$$

Then $\mathcal{K}_{\mathcal{B},\Phi}$ is a Banach subspace of K. We will call this space the atomic decomposition space of K.

Remark 7. The function Φ has no particular role except to show that many atomic decompositions can be found, by replacing b_i with $\Phi(b_i)$ provided Φ is bounded.

The next result is due to F.F. Bonsall [4], and gives conditions under which a Banach space may be norm-equivalent to its atomic decomposition space.

Proposition 8 ([4],Theorem 1)**.** *Suppose the assumptions of Theorem 6 are satisfied. If there exists an absolute constant C such that*

$$\sup_{i\in I} |\psi(\Phi(b_i))| \geq C\|\psi\|_{K^*}, \quad \text{for all } \psi \in K^*,$$

then

$$\|f\|_K \cong \|f\|_{\mathcal{K}_{\mathcal{B},\Phi}}.$$

Remark 9. It will be important to note that the two constants involved in the conclusion of Proposition (8) are $C_1 = \sup_{i\in I} \|b_i\|_K$ and $C_2 = C$.

The next result is about providing a candidate for the atomic decomposition of Lorentz-Bochner spaces $L_{p,q}$. For simplicity, we will denote χ_{Ω_n} (Where Ω_n is given such as in (1)) by b_n for all $n \in \mathbb{N}$.

Lemma 10. *Let $1 \leq q \leq \infty, 1 < p \leq \infty$, and Ω be given as in* (1)*. Consider $B^{p,q}$ the set of strongly measurable functions $f : \Omega \to X$ such that*

$$f = \sum_{n=1}^{\infty} x_n^{p,q} b_n, \quad \text{with } x_n^{p,q} \in X \text{ , and } \{x_n^{p,q}\}_{n\in\mathbb{N}} \in l^1(X).$$

Then $(B^{p,q}, \|\cdot\|_{B^{p,q}})$ is a Banach subspace, dense in $L_{p,q}$ with

$$\|f\|_{B^{p,q}} = \inf \sum_{n=1}^{\infty} \|x_n^{p,q}\|, \text{ where the infimum is taken over all representations of } f.$$

Remark 11. Note that the space $B^{p,q}$ was first introduced in the early eighties by De Souza in [6], for $\Omega = [0, 2\pi]$ and $X = \mathbb{C},\ q = p$ and was denoted by B^p, the space formed by special atoms.

Theorem 12. *Let $1 < p < \infty,\ 1 \leq q < \infty$. Suppose $C(p, q) = \sup_{f\in L_{p,q}} \|f\|_{B^{p,q}} < \infty$. Then*

$$|||f\|_{p,q} \cong \|f\|_{B^{p,q}}, \quad \text{for all } f \in L_{p,q}.$$

Remark 13. This theorem generalizes the result obtained by De Souza in [7] for $q = 1,\ \Omega = [0, 2\pi]$ and $X = \mathbb{C}$.

Corollary 14. *Let T be a continuous operator on $L_{p,q}$ not necessarily linear. If T is bounded on $B^{p,q}$, then T is bounded on $L_{p,q}$.*

3 Applications

Let $1 < p < \infty, \quad 1 \leq q < \infty$. We will study the boundedness of some operators on $L_{p,q}$ and $B^{p,q}$. Throughout this section, all the constants will be generic constants for simplicity. For a function $f \in B^{p,q}$, we consider it as $f = \sum_{n=1}^{\infty} x_n^{p,q} b_n$ with $\sum_{n=1}^{\infty} |x_n^{p,q}| < \infty$.

3.1 Centered Hardy-Littlewood Maximal Operator

Corollary 15. *For $X = \mathbb{C}$, the centered Hardy-Littlewood maximal M operator maps $B^{p,q}$ to $B^{p,q}$ boundedly.*

3.2 Hilbert Transform

In the next result, our concern is the boundedness of the Hilbert transform H on $L_{p,q}$. In the literature, the boundedness is obtained (see [3]) using the following inequality

$$(Hf)^*(t) \leq C\left(\frac{q}{t}\int_0^t f*(s)ds + \int_t^\infty f^*(s)\frac{ds}{s}\right), \text{ for some positive constant } C.$$

We show that, under equivalent Sawyer's-type condition (3) for $L_{p,q}$ spaces (see [16]), the atomic decomposition can also be used to obtain the proof of the boundedness of H on $L_{p,q}$. Note that similar assumptions are used by Nazarov et al. (see [15], Theorem 2.1) to show the boundedness of H on L_2.

Corollary 16. *Let $\Omega = X = \mathbb{R}$. Suppose there is a universal constant such K such that for any interval A,*

$$\|H\chi_A\|_{p,q} \leq K\mu(A)^{1/p}. \tag{3}$$

Then the Hilbert transform $H : L_{p,q} \to L_{p,q}$ is bounded.

Remark 17. Michael Lacey rightfully pointed to the author during a correspondence that in the case of the Hilbert operator, the atoms b_n have to be characteristic functions of intervals, unless they are chosen so that $\|b_n\|_{p,q} < \|\chi_{\Omega_n}\|_{p,q}$.

Proofs of the next two results can also be found in [1], [2], and [10]. When applying the atomic decomposition on $L_{p,q}$, these results are obtained easily.

3.3 Multiplication Operator

Corollary 18. *Let $g : \Omega \to \mathcal{B}(X)$ be a strongly measurable function. Then the multiplication operator $M_g : L_{p,q} \to L_{p,q}, \quad f \mapsto M_g f = f \cdot g$ is a bounded operator if $g \in L^\infty(\Omega, \mathcal{B}(X))$.*

3.4 Composition Operator

Corollary 19. *Let $h : \Omega \to \Omega$ be a non-singular measurable transformation, then the composition operator $C_h : L_{p,q} \to L_{p,q}, \quad f \mapsto C_h f = f \circ h$ is a bounded operator on $L_{p,q}$ if there is an absolute constant M such that $\mu(h^{-1}(A)) < M\mu(A)$, for all $A \in \mathcal{A}$.*

3.5 Interpolation Theorem

Corollary 20. *Let T be a continuous operator of restricted-type (p_i, q_i) with $1 < p_i, q_i < \infty,\ i = 0, 1$. Put*

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1} \quad \text{for some } \theta \in (0,1). \tag{4}$$

Then the operator T is of type (p, q).

4 Concluding remarks

The atomic decomposition we propose in this paper has some nice applications. Though we study just the Hardy, the Hilbert, the composition and multiplication operators, we believe our result could be applied to more than just these operators. In fact, Michael Lacey [11] who intensively studied the Carleson operator pointed to us that the results obtained here can be used to study Carleson operators on Fourier series on $L_{p,q}$ spaces. Going forward, it would be interesting to see how we can generalize these results to Calderon-Zygmund operators in order to obtain the a T_1-type theorem.

Acknowledgment. The author wishes to thank Micheal Lacey for the valuable comments and insights to the paper.

References

[1] S. C. Arora, G. Datt and S. Verma, *Composition Operators on Lorentz Spaces*, Bull. Austral. Math. Soc. **76** (2007) 205–214.

[2] S. C. Arora, G. Datt and S. Verma, *Multplication and Composition Operators on Lorentz-Bochner Spaces* Osaka J. Math. **45** (2008), 629-641.

[3] C. Bennet, R. Sharley *Interpolation Operators* Academic Press., (1988).

[4] F.F. Bonsall , *Decomposition of Functions as Sums of Elementary Functions* Quart. J. Math, Oxford Ser. **2** (1986), 355-365.

[5] R. R. Coifman, *A Real Variable Characterization of H^p* , Studia Math. **51** (1974), 241-250.

[6] G. De Souza, *Spaces Formed by Special Atoms*, PhD dissertation, SUNY at Albany, 1980.

[7] G. De Souza, *A Proof of Carleson Theorem Based on a New Characterization of Lorentz Spaces $L(p, 1)$ and Other Applications*, to appear.

[8] C. Fefferman, *Characterization of Bounded Mean Oscillation*, Bull. A.M.S, **77**, (1971), 587-588.

[9] H. Feichtinger, G. Zimmerman, *An exotic Minimal Banach Space of Functions*, Math, Nachr. **239-240**, (2002), 42-61.

[10] E. Kwessi, P. Alfonso, G. De Souza, A. Abebe, *A Note on Multiplication and Composition Operators in Lorentz Spaces*, Journal of Functions Spaces and Applications, Vol. 2012, Article ID 293613, 10 pages, 2012. doi:10.1155/2012/293613.

[11] M. T. Lacey, C. Thiele, *A proof of boundedness of the Carleson operator*, Math. Res. Lett. 7 (2000), no. 4, 361370.

[12] G. G. Lorentz, *Some New Function Spaces*, Ann. of Math. **51** (1950), 37-55.

[13] G. G. Lorentz, *On the theory of spaces* , Pacific Journal of Mathematics **1** (1951), 411-429.

[14] P.D. Liu, Y. L. Hou, *Atomic Decomposition of Banach-Spaces-Valued Martingales*, sci in China, Serieces A Math. **42** (1999), 38-47.

[15] F. Nazarov, S, treil, A. Volberg, *Two Weight Estimate for the Hilbert Transform and Corona Decomposition for Non-doubilng Measures*, arXiv:1003.1596v1 [math.AP]

[16] E. Sawyer, *Boundedness of classical operators on classical Lorentzs spaces*, Studia math. **96** (1990), 145-158.

[17] F. Weisz, *Martingales Hardy Spaces and Their Applications in Fourrier Analysis* , Lecture Notes in Math **1568**, New York: Springer-Verlag, 1994.

[18] J. Yong, P. Linua, L. Peide, *Atomic Decomposition of Lorentz Martingales spaces and Applications*, J. Func. Spaces and Applications, **7**, 2 (2009), 153-166.