Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 626314, 9 pages
http://dx.doi.org/10.1155/2014/626314

Research Article

Atomic Decomposition of Weighted Lorentz Spaces and Operators

Eddy Kwessi,[1] Geraldo de Souza,[2] Fidele Ngwane,[3] and Asheber Abebe[2]

[1] *Department of Mathematics, Trinity University, 1 Trinity Place, San Antonio, TX 78212, USA*
[2] *Department of Mathematics and Statistics, Auburn University, 221 Parker Hall, Auburn, AL 36849, USA*
[3] *Department of Sciences and Mathematics, University of South Carolina Salkehatchie, 807 Hampton Street, Walterboro, SC 24988, USA*

Correspondence should be addressed to Eddy Kwessi; ekwessi@trinity.edu

Received 2 February 2014; Accepted 11 April 2014

Academic Editor: Qingying Bu

We obtain an atomic decomposition of weighted Lorentz spaces for a class of weights satisfying the Δ_2 condition. Consequently, we study operators such as the multiplication and composition operators and also provide Hölder's-type and duality-Riesz type inequalities on these weighted Lorentz spaces.

1. Introduction

Weighted spaces are studied in most cases as a generalization of a special case. The Lorentz spaces, introduced by Lorentz in [1, 2], are no exception to this. The first version of the weighted Lorentz spaces was provided by Lorentz himself and was defined as $\Lambda^p(w) = \{f : \mathbb{R}^N \to \mathbb{R} : \|f\|_{\Lambda^p(w)} = (\int_0^\infty (f^*(x))^p w(x)dx)^{1/p} < \infty\}$, where f^* is the decreasing rearrangement of f and w is a weight function. He proved that, for $p \geq 1$, $\|\cdot\|_{\Lambda^p(w)}$ is a norm if and only if the weight w is decreasing. Carro and Soria in [3] proved that $\|\cdot\|_{\Lambda^p(w)}$ is a quasi-norm in general provided that $W(t) = \int_0^s w(s)ds$ satisfies the Δ_2 condition, that is, $W(2t) \leq CW(t)$, for some constant $C > 1$. In this paper, we study $\Lambda^1(w)$ that we denote by L_Φ, with $\Phi(t) = tw(t)$, in the sense that $f : [0, 2\pi] \to \mathbb{R}$ belongs to L_Φ if and only if $\|f\|_{L_\Phi} = \int_0^{2\pi} f^*(t)(\Phi(t)/t)dt < \infty$. Our interest in this special space stems from the fact that, as demonstrated in [4] with $w(t) = t^{p/q}$, this space has some interesting properties that allow an easy study of operators on $L(p, q)$ via the Interpolation Theorem. The atomic decomposition of Banach spaces has been studied by many authors before: the Fourier transform of a function over the space $L^2[0, 2\pi]$ can be thought of as an atomic decomposition of the space $L^2[0, 2\pi]$. Coifman in [5] gave the unifying definition of an atom and showed that Hardy's spaces $H^1(\mathbb{D})$, the spaces of holomorphic functions on the unit disc $\mathbb{D}$, have an atomic decomposition and he used the latter result to prove that the dual spaces of these spaces are equivalent to the spaces of functions of bounded means oscillations. In [6], Jiao et al. proved that the Lorentz-Matingales spaces also have an atomic decomposition. In an attempt to give a different proof of the acclaimed Carleson Theorem (see e.g., [7]), de Souza [8] showed that the Lorentz spaces $L(p, 1)$ have an atomic decomposition. In this paper, we continue the ideas in [8] and show that the weighted Lorentz spaces also admit an atomic decomposition, for a certain class of weights.

The remainder of the paper is organized as follows. In the preliminaries section, we introduce the necessary notions needed; namely, we define the conditions on our weight functions, and provide some preliminary definitions and results. In the second section, we prove that the weighted Lorentz spaces have an atomic decomposition and in the third section, we utilize this atomic decomposition to show the boundedness of some operators on theses weighted spaces. The last section opens up a discussion about the relevance of this line of research.

2. Preliminaries

We begin with some preliminary definitions and results (proofs can be found in the appendices) that will be helpful throughout the paper.

Definition 1. Define C_Φ as the space of weights $\Phi : [0,\infty) \to [0,\infty)$ so that,

(1) $\Phi(0) = 0$,

(2) Φ is increasing,

(3) $\Phi(t)/t$ is decreasing,

(4) there is a positive constant C such that, $\int_0^x (\Phi(t)/t) \leq C\Phi(x)$, (Dini's Condition),

(5) Φ satisfies the Δ_2 condition; that is, there is a an constant $K > 1$ such that $\Phi(2x) \leq K\Phi(x)$.

Note that the space C_Φ is nonempty since $\Phi(t) = t^\alpha$, for $\alpha \in (0,1)$, belongs to C_Φ. Hereafter, μ will denote a nonatomic measure defined on $[0,2\pi]$.

Definition 2. One defines the weighted Lorentz space L_Φ as

$$L_\Phi = \left\{ f : [0,2\pi] \longrightarrow \mathbb{R}; \|f\|_{L_\Phi} = \int_0^{2\pi} f^*(t) \frac{\Phi(t)}{t} dt < \infty \right\}, \tag{1}$$

where f^* is the decreasing rearrangement of f defined as $f^*(t) = \inf\{y > 0 : \mu(\{x : |f(x)| > y\}) \leq t\}$, and μ is a nonatomic measure defined on $[0,2\pi]$.

Remark 3. For $\Phi(t) = t^{1/p}$, L_Φ is identical to the classical Lorentz space $L(p,1)$.

Definition 4. One will also consider the following space:

$$A_\Phi(\mu) = \left\{ f : [0,2\pi] \longrightarrow \mathbb{R}, f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}; \right. \\ \left. \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) < \infty \right\}, \tag{2}$$

where the A_n's are μ-measurable sets in $[0,2\pi]$ and χ_A represents the characteristic function on the set A.

We will show in Theorem 14 that this space is an atomic decomposition of the space L_Φ.

Put $\|f\|_{A_\Phi(\mu)} = \inf \sum_{n=1}^{\infty} |c_n|\Phi(\mu(A_n))$ where the infimum is taken over all possible representations of f. The next result is proved in the appendix.

Proposition 5. *If one endows $A_\Phi(\mu)$ with $\|\cdot\|_{A_\Phi(\mu)}$, then*

(1) *$\|\cdot\|_{A_\Phi(\mu)}$ is a norm,*

(2) *$(A_\Phi(\mu), \|\cdot\|_{A_\Phi(\mu)})$ is a Banach space.*

Definition 6. For $1 \leq r \leq \infty$, define for a measurable function $g : [0,2\pi] \to \mathbb{R}$ the quantity

$$\|g\|_{M_{\Phi,r}} = \begin{cases} \sup_{x>0} \left(\frac{1}{\Phi(x)} \int_0^x (g^*(t)\Phi(t))^r \frac{dt}{t} \right)^{1/r} & \text{if } 1 \leq r < \infty, \\ \sup_{x>0} g^*(x)\Phi(x) & \text{if } r = \infty. \end{cases} \tag{3}$$

The space $M_{\Phi,r}$ is the set of measurable functions g for which $\|g\|_{M_{\Phi,r}} < \infty$. This space generalizes the space M_r^p introduced in [4]. In the next theorem and remark, we give further properties of theses spaces in the weighted case.

Theorem 7. *For $\Phi \in C_\Phi$, one has that*

(1) *if $1 \leq r \leq \infty$, then $\|\cdot\|_{M_{\Phi,r}}$ is a quasi-norm on $M_{\Phi,r}$,*

(2) *$L_\infty \cong M_{\Phi,1}$ with $\|g\|_\infty \cong \|g\|_{M_{\Phi,1}}$.*

The remark below is stated only for completeness and the proof can be found in [4].

Remark 8. For $\Phi \in C_\Phi$ and $r > 1$, we have $M_{\Phi,r} \cong M_{\Phi^{1/r'},\infty}$, where $1/r + 1/r' = 1$.

Definition 9. For $\Phi \in C_\Phi$, define $\Psi(t) = t/\Phi(t)$, $t > 0$. For a measure μ defined on $[0,2\pi]$, consider the following spaces

$$\Sigma_\Phi^1(\mu) = \left\{ g : [0,2\pi] \longrightarrow \mathbb{R} \text{ measurable} : \right. \\ \left. \|g\|_{\Sigma_\Phi^1(\mu)} = \sup_{\mu(A)\neq 0} \frac{1}{\Phi(\mu(A))} \left| \int_A g(t)\, d\mu(t) \right| < \infty \right\},$$
$$L_\Psi^\infty = \left\{ g : [0,2\pi] \longrightarrow \mathbb{R} \text{ measurable} : \right. \\ \left. \|g\|_{L_\Psi^\infty} = \sup_{t>0} \Psi(t)\, g^*(t) < \infty \right\}. \tag{4}$$

The first space was basically introduced by Lorentz in [2] for $\Phi(t) = t^{1/p}$. We will prove in Theorem 10 that these spaces are norm-equivalent.

Theorem 10. *Let $\Phi \in C_\Phi$ and $\Psi(t) = t/\Phi(t)$, $t > 0$. For a measurable function $g : [0,2\pi] \to \mathbb{R}$, one has*

$$\|g\|_{\Sigma_\Phi^1(\mu)} \cong \|g\|_{L_\Psi^\infty}. \tag{5}$$

Theorem 11 (Hölder's type inequalities). *Let $\Phi \in C_\Phi$ and $\Psi(t) = t/\Phi(t)$, $t > 0$.*

(1) *For $f \in L_\Phi$ and $g \in L_\Psi^\infty$, one has*

$$\left| \int_0^{2\pi} f(t)\, g(t)\, d\mu(t) \right| \leq \|f\|_{L_\Phi} \|g\|_{L_\Psi^\infty}. \tag{6}$$

(2) *For* $f \in A_\Phi(\mu)$ *and* $g \in \Sigma^1_\Phi(\mu)$, *one has*

$$\left|\int_0^{2\pi} f(t)\, g(t)\, d\mu(t)\right| \le \|f\|_{A_\Phi(\mu)} \|g\|_{\Sigma^1_\Phi(\mu)}. \quad (7)$$

3. Atomic Decomposition

We start with this important result on the dual of the spaces L_Φ and $A_\Phi(\mu)$.

Theorem 12. *Let* $\Phi \in C_\Phi$ *and* $\Psi(t) = t/\Phi(t)$, $t > 0$. *Then one has the following.*

(1) $(L_\Phi)^* \cong L^\infty_\Psi$; *that is,* $\phi \in (L_\Phi)^*$ *if and only if there is a unique* $g \in L^\infty_\Psi$ *so that for all* $f \in L_\Phi$

$$\phi(f) = \int_0^{2\pi} f(t)\, g(t)\, d\mu(t) \quad \text{with } \|\phi\| \cong \|g\|_{L^\infty_\Psi}. \quad (8)$$

(2) *Likewise, one has* $(A_\Phi(\mu))^* \cong \Sigma^1_\Phi(\mu)$; *that is,* $\psi \in (A_\Phi(\mu))^*$ *if and only if there is a unique* $g \in \Sigma^1_\Phi(\mu)$ *so that for all* $f \in A_\Phi(\mu)$

$$\psi(f) = \int_0^{2\pi} f(t)\, g(t)\, d\mu, \quad \|\psi\| = \|g\|_{\Sigma^1_\Phi(\mu)}. \quad (9)$$

Proof. Let $g \in L^\infty_\Psi$. Define $\phi(f) = \int_0^{2\pi} f(t)g(t)d\mu(t)$. By Theorem 11, we have

$$|\phi(f)| \le \|f\|_{L_\Phi} \|g\|_{L^\infty_\Psi}. \quad (10)$$

Thus, using the linearity of the integral, we conclude that $\phi \in (L_\Phi)^*$. On the other hand, let $\phi \in (L_\Phi)^*$. For a μ-measurable set $A \subseteq [0, 2\pi]$, define $\lambda(A) = \phi(\chi_A)$. Then there is a constant $M > 0$ such that

$$|\lambda(A)| = |\phi(\chi_A)| \le M \|\chi_A\|_{L_\Phi}. \quad (11)$$

Since

$$\|\chi_A\|_{L_\Phi} = \int_0^{2\pi} \chi_{[0,\mu(A)]}(t) \frac{\Phi(t)}{t} dt = \int_0^{\mu(A)} \frac{\Phi(t)}{t}, \quad (12)$$

then using Dini's condition (4) in Definition 1, we have

$$\|\chi_A\|_{L_\Phi} \le C\Phi(\mu(A)). \quad (13)$$

It follows from (11) and (13) that $|\lambda(A)| < MC\Phi(\mu(\mathrm{A}))$ and condition 1 in Definition 1 yield $\lambda \ll \mu$. By the Radon-Nikodym Theorem and the definition of functions in L_Φ, there is an integrable function g on $[0, 2\pi]$ such that, for all $f \in L_\Phi$,

$$\phi(f) = \int_0^{2\pi} f(t)\, g(t)\, d\mu(t). \quad (14)$$

To prove that $g \in L^\infty_\Psi$, observe that

$$\left|\int_A g(t)\, d\mu(t)\right| = |\phi(\chi_A)| \le M \|\chi_A\|_{L_\Phi} < MC\Phi(\mu(A)). \quad (15)$$

Thus taking the supremum over μ-measurable sets A such that $\mu(A) \ne 0$, we have

$$\sup_{\mu(A)\ne 0} \frac{1}{\Phi(\mu(A))} \left|\int_A g(t)\, d\mu(t)\right| \le MC, \quad \text{that is, } g \in \sum_\Phi^1(\mu). \quad (16)$$

The proof is complete using the equivalence in Theorem 10. The proof of the second part is very similar to that of the first part and uses the second part of Theorem 12. □

The following result is a classical result in function analysis. (See e.g., [9, page 160], for a proof.)

Theorem 13. *Let* X *and* Y *be two vector normed spaces and let* $T \in L(X, Y)$, *the space of bounded linear operators from* X *onto* Y. *Let* T^* *be the adjoint operator of* T *defined by* $T^* f = f \circ T$ *for all* $f \in Y^*$, *the dual space of* Y. *Then*

(a) $T^* \in L(Y^*, X^*)$ *and* $\|T^*\| = \|T\|$;

(b) T^* *is injective if and only if the range of* T *is dense in* Y.

The next result is the most important of the present paper and gives an equivalent representation of functions in L_Φ as "linear" combinations of simple functions.

Theorem 14 (atomic decomposition of L_Φ). *For* $\Phi \in C_\Phi$, *one has*

$$L_\Phi \cong A_\Phi(\mu), \quad \text{with } \|f\|_{L_\Phi} \cong \|f\|_{A_\Phi(\mu)}. \quad (17)$$

Proof. Let us show first that $A_\Phi(\mu) \subseteq L_\Phi$. Take $f(t) = \chi_A(t)$. Then using Dini's condition (4) in Definition 1, we have

$$\|\chi_A\|_{L_\Phi} = \int_0^{2\pi} \chi_A^*(t) \frac{\Phi(t)}{t} dt = \int_0^{\mu(A)} \frac{\Phi(t)}{t} dt \le C\Phi(\mu(A)). \quad (18)$$

Thus if $f(t) = \sum_{n=1}^\infty c_n \chi_{A_n}(t)$ for $c_n \in \mathbb{R}$, then

$$\|f\|_{L_\Phi} \le \sum_{n=1}^\infty |c_n| \left\|\chi_{A_n}\right\|_{L_\Phi}. \quad (19)$$

And (18) implies

$$\|f\|_{L_\Phi} \le C \sum_{n=1}^\infty |c_n| \Phi(\mu(A)). \quad (20)$$

Taking the infimum over all representations of f, we have

$$\|f\|_{L_\Phi} \le C \|f\|_{A_\Phi(\mu)}, \quad \text{that is, } A_\Phi(\mu) \subseteq L_\Phi. \quad (21)$$

To prove the other direction, we can use either Theorem 1 in [10] or Theorem 13. In this paper, we will use the latter. Note that we have the following:

A_1: $A_\Phi(\mu) \subseteq L_\Phi$ and $\|f\|_{L_\Phi} \le M \|f\|_{A_\Phi(\mu)}$, by inequality (21).

A_2: $A_\Phi(\mu)$ is dense in L_Φ, see [11].

A_3: $(A_\Phi(\mu))^* = (L_\Phi)^*$ since by Theorems 10 and 12 $(L_\Phi)^* \cong L_\Psi^\infty \cong \Sigma_\Phi^1(\mu) \cong (A_\Phi(\mu))^*$.

Using Theorem 13, we conclude from A_1 that the inclusion map $i : A_\Phi(\mu) \to L_\Phi$ is a bounded linear map and that $\|i^*\| = \|i\|$ where i^* is the duality map $i^* : (L_\Phi)^* \to (A_\Phi(\mu))^*$. From A_2, it follows that duality map i^* is injective and from A_3 that i^* is the identity map. Therefore, we have that i is an isomorphism and thus $\|f\|_{L_\Phi} \cong \|f\|_{A_\Phi(\mu)}$. □

Remark 15. The space $A_\Phi(\mu)$ is called an atomic decomposition of L_Φ in the sense that each function of L_Φ coincides with a function of $A_\Phi(\mu)$ and thus can be written as a "linear" combination of atoms, where the atoms are the "simple" functions χ_{A_n}.

4. Operators on Weighted Lorentz Spaces

In this section, we study two types of operators: the multiplication and composition operators of weighted Lorentz spaces L_Φ.

Theorem 16 (multiplication operator). *For $\Phi \in C_\Phi$ and for $f \in L_\Phi$, define the multiplication operator T_g as $T_g f = f \cdot g$. Then $T_g : L_\Phi \to L_\Phi$ is bounded if and only if $g \in M_{\Phi,1}$. Moreover, $\|T_g\| \cong \|g\|_{M_{\Phi,1}}$.*

Proof. If T_g is bounded, then there is an absolute constant M such that

$$\left\|T_g f\right\|_{L_\Phi} \le M\|f\|_{L_\Phi}, \quad \forall f \in L_\Phi. \tag{22}$$

Take $f = \chi_A$ for $A \subseteq [0, 2\pi]$. Then from (22), it follows that

$$\int_0^{2\pi} \left(\chi_A(t)\, g(t)\right)^* \frac{\Phi(t)}{t} dt \le M \int_0^{2\pi} \chi_A^*(t) \frac{\Phi(t)}{t} dt, \tag{23}$$

which after simplification is equivalent to

$$\int_0^{\mu(A)} g^*(t) \frac{\Phi(t)}{t} dt \le M \int_0^{\mu(A)} \frac{\Phi(t)}{t} dt. \tag{24}$$

Using Dini's condition (4) in Definition 1, we have

$$\int_0^{\mu(A)} g^*(t) \frac{\Phi(t)}{t} dt \le MC\Phi\left(\mu(A)\right), \tag{25}$$

and hence

$$\frac{1}{\Phi\left(\mu(A)\right)} \int_0^{\mu(A)} g^*(t) \frac{\Phi(t)}{t} dt \le K. \tag{26}$$

Thus,

$$\sup_{x>0,\mu(A)=x} \frac{1}{\Phi(x)} \int_0^{x} g^*(t) \frac{\Phi(t)}{t} dt \le K. \tag{27}$$

This completes the proof that $g \in M_{\Phi,1}$.

On the other hand, if $g \in M_{\Phi,1}$, then for $A \subseteq [0, 2\pi]$, we have

$$\begin{aligned}\left\|T_g \chi_A\right\|_{L_\Phi} &= \int_0^{2\pi} \left(\chi_A \cdot g\right)^*(t) \frac{\Phi(t)}{t} dt \\ &= \int_0^{\mu(A)} g^*(t) \frac{\Phi(t)}{t} dt \\ &= \Phi\left(\mu(A)\right)\left(\frac{1}{\Phi\left(\mu(A)\right)} \int_0^{\mu(A)} g^*(t) \frac{\Phi(t)}{t} dt\right).\end{aligned} \tag{28}$$

Therefore,

$$\left\|T_g \chi_A\right\|_{L_\Phi} \le \|g\|_{M_{\Phi,1}} \cdot \Phi\left(\mu(A)\right). \tag{29}$$

Since from Theorem 14, $L_\Phi \cong A_\Phi(\mu)$; then for $f \in L_\Phi$, we have $f(t) = \sum_{n=1}^{\infty} c_n \Phi(\mu(A_n))$, for some c_n's $\in \mathbb{R}$. And so,

$$\left\|T_g f\right\|_{L_\Phi} \le \sum_{n=1}^{\infty} \left|c_n\right| \left\|T_g \chi_{A_n}\right\|_{L_\Phi} \le \|g\|_{M_{\Phi,1}} \sum_{n=1}^{\infty} \left|c_n\right| \Phi\left(\mu\left(A_n\right)\right). \tag{30}$$

Taking the infimum over all representations of f and using the equivalence between L_Φ and $A_\Phi(\mu)$, we have

$$\left\|T_g f\right\|_{L_\Phi} \le \|g\|_{M_{\Phi,1}} \|f\|_{L_\Phi}. \tag{31}$$

To prove the second statement of the theorem, observe that (31) implies that

$$\left\|T_g\right\| \le \|g\|_{M_{\Phi,1}}. \tag{32}$$

If we take $f(t) = (1/\Phi(x))\chi_{[0,x]}(t)$ for some $x > 0$, then $\|f\|_{L_\Phi} = 1$ and since $g^*(t)$ and $\Phi(t)/t$ are decreasing, we have

$$\begin{aligned}\left\|T_g f\right\|_{L_\Phi} &= \frac{1}{\Phi(x)} \int_0^{2\pi} \left(\chi_{[0,x]}\right)^*(t) \frac{\Phi(t)}{t} dt \\ &= \frac{1}{\Phi(x)} \int_0^{x} g^*(t) \frac{\Phi(t)}{t} dt \\ &\ge \frac{1}{\Phi(x)} g^*(x) \int_0^{x} \frac{\Phi(t)}{t} dt \ge g^*(x).\end{aligned} \tag{33}$$

Consequently, $\|T_g\|_{L_\Phi} \ge g^*(x)$ and so $\sup_{\|f\|_{L_\Phi}=1} \|T_g f\|_{L_\Phi} \ge g^*(x)$. Thus

$$\left\|T_g\right\| \ge g^*(x), \quad \forall x > 0. \tag{34}$$

Taking the limit as $x \to 0$, we have

$$\left\|T_g\right\| \ge \|g\|_\infty. \tag{35}$$

From the inequality (B.4) in the proof of Theorem 7 (Appendix B), it follows that

$$\left\|T_g\right\| \ge C\|g\|_{M_{\Phi,1}}. \tag{36}$$

The result then follows by combining (32) and (36). □

Definition 17. Let h be a nonsingular measurable transformation on $[0, 2\pi]$, and let $\Phi \in C_\Phi$. We define $X_\Phi(\mu)$ as

$$X_\Phi(\mu) = \{h : [0, 2\pi] \longrightarrow [0, 2\pi] : \Phi(\mu(h^{-1}(A))) \le M\Phi(\mu(A))\}, \quad (37)$$

where A is a μ-measurable set in $[0, 2\pi]$. M is an absolute real constant and h^{-1} is the inverse image of the μ-measurable subset A of $[0, 2\pi]$. Put $\|h\|_{X_\Phi(\mu)} = \sup_{\mu(A)\neq 0}(\Phi(\mu(h^{-1}(A)))/\Phi(\mu(A)))$.

Theorem 18 (composition operator). *For $\Phi \in C_\Phi$ and for $f \in L_\Phi$, define the composition operator C_h as $C_h f = f \circ h$. Then $C_h : L_\Phi \to L_\Phi$ is bounded if and only if $h \in X_\Phi(\mu)$. Moreover, $\|C_h\| \cong \|h\|_{X_\Phi(\mu)}$.*

Proof. The technique of this proof mirrors that of Theorem 16. For, assume that the operator C_h is bounded; that is, there is some absolute constant M such that

$$\|C_h f\|_{L_\Phi} \le M\|f\|_{L_\Phi}. \quad (38)$$

Taking $f = \chi_A$ for some μ-measurable set $A \subseteq [0, 2\pi]$, the inequality (38) implies

$$\int_0^{2\pi} (\chi_A \circ h)^*(t) \frac{\Phi(t)}{t} dt \le M \int_0^{2\pi} \chi_{[0,\mu(A)]}(t) \frac{\Phi(t)}{t} dt. \quad (39)$$

Since $(\chi_A \circ h)(t) = \chi_{h^{-1}(A)}(t)$, (39) entails

$$\int_0^{\mu(h^{-1}(A))} \frac{\Phi(t)}{t} dt \le M \int_0^{\mu(A)} \frac{\Phi(t)}{t} dt. \quad (40)$$

Using the fact that $\Phi(t)/t$ is decreasing and Dini's condition 4 in Definition 1, respectively, on the LHS and the RHS of (40), we have

$$\Phi(\mu(h^{-1}(A))) \le M\Phi(\mu(A)). \quad (41)$$

This proves that $h \in X_\Phi(\mu)$.

On the other hand, suppose that $h \in X_\Phi(\mu)$. Then for a μ-measurable subset A of $[0, 2\pi]$ we have

$$\|C_h \chi_A\|_{L_\Phi} = \int_0^{\mu(h^{-1}(A))} \frac{\Phi(t)}{t} dt. \quad (42)$$

Using Dini's condition (4) and the fact that $h \in X_\Phi(\mu)$, there is an absolute constant M such that

$$\|C_h \chi_A\|_{L_\Phi} \le C\Phi(\mu(h^{-1}(A))) \le CM\Phi(\mu(A)). \quad (43)$$

Now let $f \in L_\Phi$. Then $f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}(t)$ since $L_\Phi \cong A_\Phi(\mu)$. Then, using (43), it follows that

$$\|C_h f\|_{L_\Phi} \le MC \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)). \quad (44)$$

Taking the infimum over all representations of f and using the equivalence $L_\Phi \cong A_\Phi(\mu)$ we have

$$\|C_h f\|_{L_\Phi} \le \|f\|_{L_\Phi}, \quad (45)$$

showing that the operator C_h is bounded on L_Φ.

Note that, from (45), we get $\|C_h\| \le MC$. Without loss of generality, consider the constant M to be such that $M = \inf\{K > 0 : K \ge \Phi(\mu(h^{-1}(A)))/\Phi(\mu(A))\}$. Then

$$\|C_h\| \le CM \le C\left(\sup_{\mu(A)\neq 0} \frac{\Phi(\mu(h^{-1}(A)))}{\Phi(\mu(A))}\right) = C\|h\|_{X_\Phi(\mu)}. \quad (46)$$

Moreover, if we take $f(t) = \chi_A(t)/\Phi(\mu(A))$ for a μ-measurable subset A in $[0, 2\pi]$, we have $\|f\|_{L_\Phi} = 1$ and using the fact that $\Phi(t)/t$ is decreasing

$$\begin{aligned}\|C_h f\|_{L_\Phi} &= \frac{1}{\Phi(\mu(A))} \int_0^{\mu(g^{-1}(A))} \frac{\Phi(t)}{t} dt \\ &\ge \frac{\Phi(\mu(h^{-1}(A)))}{\Phi(\mu(A))}.\end{aligned} \quad (47)$$

Hence

$$\|C_h\| = \sup_{\|f\|_{L_\Phi}=1} \|C_h f\|_{L_\Phi} \ge \frac{\Phi(\mu(h^{-1}(A)))}{\Phi(\mu(A))}. \quad (48)$$

Taking the supremum over $A \subseteq [0, 2\pi]$ such that $\mu(A) \neq 0$, we have that

$$\|C_h\| \ge \|h\|_{X_\Phi(\mu)}. \quad (49)$$

The proof of the second part is complete by combining (46) and (49). □

Remark 19. The previous result in part shows that boundedness of operators other than the aforementioned ones on weighted Lorentz spaces $L_\Phi(\mu)$ is possible if their action on characteristic functions can be controlled. In particular, the centered Hardy-Littlewood Maximal operator, the Hilbert operator (under Sawyer's type condition) are bounded on L_Φ.

5. Discussion

The special atoms spaces $A_\Phi(\mu)$ originally introduced by de Souza in [12] for $\Phi(t) = t^{1/p}$ seem to have an interesting role in analysis with its connection to Lipschitz spaces (see [13]) through Hölder's inequality and duality. These spaces allow for simple characterization of the Bergman-Besov-Lipschitz spaces (see [11]), that is, spaces of functions f defined on $[0, 2\pi]$ such that

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(x) - f(y)|}{|x - y|^{2-1/p}} dx\, dy < \infty, \quad p > 1. \quad (50)$$

Another interesting use of the special atoms space is the real characterization of some spaces of analytic functions F in the unit disc such that

$$\int_0^1 \int_0^{2\pi} \left|F'\left(re^{i\theta}\right)\right| (1-r)^{(1/p)-1} d\theta dt < \infty, \quad p > 1, \tag{51}$$

where F' represents the derivative of F (see [11, 14]). The special atom spaces have been generalized in a couple of different ways: one is the weighted case with its connections to weighted Lipschitz spaces and other weighted spaces of analytic functions. The other is that, unlike in the original definition of special atoms spaces where the atoms were intervals, the atoms can now be replaced with measurable sets for general measures. This last generalization has led to the study of Lorentz spaces $L(p,1)$, $p > 1$ and the weak-L_p spaces also known as $L(p,\infty)$, $p > 1$. Indeed in [8], we show that $f \in L(p,1)$ for $p > 1$ if and only if

$$f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}(t), \tag{52}$$

where $\sum_{n=1}^{\infty} |c_n| \mu(A)^{1/p} < \infty$, the A_n's are μ-measurable sets in $[0,2\pi]$. It was also shown in [8] that (52) is equivalent to

$$f(t) = \sum_{n=1}^{\infty} c_n \left[\chi_{A_n}(t) - \chi_{B_n}(t)\right], \tag{53}$$

where $\sum_{n=1}^{\infty} |c_n| \mu(A_n \cup B_n)^{1/p} < \infty$ and $A_n \cap B_n = \emptyset$ for μ-measurable set A_n, B_n in $[0,2\pi]$.

What makes (52) and (53) remarkable is that they help to prove and generalize a result by Weiss and Stein ([15]) which states that a linear operator $T : L(p,1) \to B$ is bounded, where B is a Banach space closed under absolute value and satisfying $\|f\|_B = \||f|\|_B$ if $\|T\chi_A\|_B \le C\mu(A)^{1/p}$, for an absolute constant C.

Another interesting observation is that the dual of $L(p,1)$ can be identified as the set of measurable functions $g : [0,2\pi] \to \mathbb{R}$ such that either of the following is satisfied, for μ-measurable subsets A, B of $[0,2\pi]$,

$$\sup_{\mu(A)\neq 0} \frac{1}{\mu(A)^{1/p}} \left|\int_A g(t)\, d\mu(t)\right| < \infty, \tag{54}$$

$$\sup_{\substack{\mu(X)\neq 0\\ X=A\cup B\\ A\cap B=\emptyset\\ \mu(A)=\mu(B)}} \frac{1}{\mu(X)^{1/p}} \left|\int_A g(t)\, d\mu(t) - \int_B g(t)\, d\mu(t)\right|. \tag{55}$$

In fact, (54) and (55) provide a natural generalization of Lipschitz spaces. Indeed in (54), letting $g(t) = f'(t)$ for a differentiable function f on $[0,2\pi]$, $A = [x, x+h]$, and μ be the Lebesgue measure yields

$$\sup_{h>0} \frac{|f(x+h) - f(x)|}{h^{1/p}} < \infty. \tag{56}$$

Also in (55), letting $g(t) = f'(t)$, $A = [x-h, x]$, $B = [x, x+h]$, and μ be the Lebesgue measure yields

$$\sup_{h>0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{(2h)^{1/p}} < \infty. \tag{57}$$

In [4], Kwessi et al. use this new representation of $L(p,1)$ to study operators such as the multiplication and composition operators on $L(p,q)$ via interpolation. The key part is to show that the study of the boundedness of such operators on $L(p,q)$ and in particular on $L(p,p) = L_p$ amounts to the study of the action of such operators on characteristic functions of sets. The present paper follows the same idea on weighted Lorentz spaces L_Φ.

Appendices

A. Proof of Proposition 5

(a) We first prove that $\|\cdot\|_{A_\Phi(\mu)}$ is a norm on $A_\Phi(\mu)$. Let $f \in A_\Phi(\mu)$ such that $f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}(t)$. Then for $\varepsilon > 0$ arbitrary, we have that

$$\|f\|_{A_\Phi(\mu)} < \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) < \|f\|_{A_\Phi(\mu)} + \varepsilon. \tag{A.1}$$

Thus

$$\|f\|_{A_\Phi(\mu)} = 0 \text{ implies } 0 < \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) < \varepsilon. \tag{A.2}$$

Since ε is arbitrary, it follows that either $c_n = 0$ or $\Phi(\mu(A_n)) = 0$, $\forall n \in \mathbb{N}$. Since $\Phi(0) = 0$ and Φ are increasing on $[0,\infty)$, it follows that $\Phi(\mu(A_n)) = 0$ is equivalent to $\mu(A_n) = 0$, $\forall n \in \mathbb{N}$. The latter implies that the A_n's are atoms of μ in $[0,2\pi]$. But since μ is nonatomic, this is impossible. Hence $\|f\|_{A_\Phi(\mu)} = 0$ implies that $c_n = 0$ which in turn implies that $f = 0$. The homogeneity condition follows directly from the fact that $(\alpha f)(t) = \sum_{n=1}^{\infty} \alpha c_n \chi_{A_n}(t)$. Now let $f, g \in A_\Phi(\mu)$ such that $f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}(t)$, $g(t) = \sum_{n=1}^{\infty} b_n \chi_{B_n}(t)$ where $\sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) < \|f\|_{A_\Phi(\mu)} + \varepsilon/2$ and $\sum_{n=1}^{\infty} |b_n| \Phi(\mu(B_n)) < \|f\|_{A_\Phi(\mu)} + \varepsilon/2$, for some arbitrary $\varepsilon > 0$. Put

$$d_n = \begin{cases} c_{n/2} & \text{if } n \text{ is even,} \\ b_{(n+1)/2} & \text{if } n \text{ is odd,} \end{cases}$$
$$D_n = \begin{cases} A_{n/2} & \text{if } n \text{ is even,} \\ B_{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases} \tag{A.3}$$

Note that we can write $(f+g)(t) = \sum_{n=1}^{\infty} d_n \chi_{D_n}(t)$ with $\sum_{n=1}^{\infty} |d_n| \Phi(\mu(D_n)) = \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) + \sum_{n=1}^{\infty} |b_n| \Phi(\mu(B_n))$. It follows that

$$\begin{aligned} \|f+g\|_{A_\Phi(\mu)} &\le \sum_{n=1}^{\infty} |c_n| \Phi(\mu(A_n)) + \sum_{n=1}^{\infty} |b_n| \Phi(\mu(B_n)) + \varepsilon \\ &\le \|f\|_{A_\Phi(\mu)} + \|g\|_{A_\Phi(\mu)} + \varepsilon. \end{aligned} \tag{A.4}$$

Since ε is arbitrary, it finishes the proof that $\|\cdot\|_{A_\Phi(\mu)}$ is a norm on $A_\Phi(\mu)$.

(b) We now prove that $(A_\Phi(\mu), \|\cdot\|_{A_\Phi(\mu)})$ is a Banach space. It suffices to show that, for any sequence $(f_m)_{m\in\mathbb{N}} \in A_\Phi(\mu)$, we have $\|\sum_{n=1}^{\infty} f_m\|_{A_\Phi(\mu)} \le \sum_{n=1}^{\infty} \|f_m\|_{A_\Phi(\mu)}$.

Let then $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions in $A_\Phi(\mu)$. Given $\varepsilon > 0$ and an integer $m \geq 1$, let c_{m_n} be a real number and let A_{m_n} be a μ-measurable set in $[0,2\pi]$ such that $f_m(t) = \sum_{n=1}^{\infty} c_{m_n}\chi_{A_{m_n}}(t)$ with $\sum_{n=1}^{\infty}|c_{m_n}|\Phi(\mu(A_{m_n})) < \|f_m\|_{A_\Phi(\mu)} + \varepsilon/2^m$. It follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m_n}\Phi\left(\mu\left(A_{m_n}\right)\right) \leq \sum_{m=1}^{\infty}\|f_m\|_{A_\Phi(\mu)} + \varepsilon. \tag{A.5}$$

Taking the infimum over all possible representations of f_m and since ε is arbitrary we get that $\|\sum_{m=1}^{\infty} f_m\|_{A_\Phi(\mu)} \leq \sum_{m=1}^{\infty}\|f_m\|_{A_\Phi(\mu)}$ and this completes the proof.

B. Proof of Theorem 7

For $1 \leq r < \infty$, $\|f\|_{M_{\Phi,r}} = 0$ implies $\forall x > 0$, $f^*\Phi = 0$ on $L^r([0,x],(dt/t))$. Therefore, since Φ is not identically zero, we have $f^* = 0$, μ-a.e. Since we can choose equivalence classes, it follows that $f = 0$. Similarly, if $r = \infty$, $\|f\|_{M_{\Phi,\infty}} = 0$ implies that $f = 0$. The homogeneity condition $\|\alpha f\|_{M_{\Phi,r}} = |\alpha|\|f\|_{M_{\Phi,r}}$ for $\alpha \in \mathbb{R}$ follows trivially from the fact that $(\alpha f)^* = |\alpha| f^*$. Finally, consider $f, g \in M_{\Phi,r}, 1 \leq r < \infty$. Since $\Phi \in C_\Phi$, we have used the properties of the decreasing rearrangement that

$$\begin{aligned}
&\|f+g\|_{M_{\Phi,r}} \\
&\quad = \sup_{x>0}\left(\frac{1}{\Phi(x)}\int_0^x \left[(f+g)^*(t)\,\Phi(t)\right]^r\frac{dt}{t}\right)^{1/r} \\
&\quad \leq \sup_{x>0}\left(\frac{1}{\Phi(x)}\int_0^x \left[f^*\left(\frac{t}{2}\right)\Phi(t) + g^*\left(\frac{t}{2}\right)\Phi(t)\right]^r\frac{dt}{t}\right)^{1/r} \\
&\quad \leq 2^{(r-1)/r}\sup_{x>0}\left[\left(\frac{1}{\Phi(x)}\int_0^{x/2}\left(f^*(u)\,\Phi(2u)\right)^r\frac{du}{u}\right)^{1/r}\right. \\
&\qquad\qquad \left. + \left(\frac{1}{\Phi(x)}\int_0^{x/2}\left(g^*(u)\,\Phi(2u)\right)^r\frac{du}{u}\right)^{1/r}\right] \\
&\quad \leq 2^{(r-1)/r}K\sup_{x>0}\left[\left(\frac{1}{\Phi(x)}\int_0^{x}\left(f^*(u)\,\Phi(u)\right)^r\frac{du}{u}\right)^{1/r}\right. \\
&\qquad\qquad \left. + \left(\frac{1}{\Phi(x)}\int_0^{x}\left(g^*(u)\,\Phi(u)\right)^r\frac{du}{u}\right)^{1/r}\right] \\
&\quad \leq 2^{(r-1)/r}K\left(\|f\|_{M_{\Phi,r}} + \|g\|_{M_{\Phi,r}}\right).
\end{aligned} \tag{B.1}$$

Likewise, for $r = \infty$, we have

$$\begin{aligned}
\|f+g\|_{M_{\Phi,\infty}} &= \sup_{x>0}(f+g)^*(x)\,\Phi(x) \\
&\leq \sup_{x>0}\left(f^*\left(\frac{x}{2}\right) + g^*\left(\frac{x}{2}\right)\right)\Phi(x) \\
&\leq \sup_{u>0} f^*(u)\,\Phi(2u) + \sup_{u>0} g^*(u)\,\Phi(2u) \\
&\leq K\left(\sup_{u>0} f^*(u)\,\Phi(u) + \sup_{u>0} g^*(u)\,\Phi(u)\right) \\
&\leq K\left(\|f\|_{M_{\Phi,\infty}} + \|g\|_{M_{\Phi,\infty}}\right).
\end{aligned} \tag{B.2}$$

This proves that $\|\cdot\|_{M_{\Phi,r}}$ is a quasi-norm on $M_{\Phi,r}$, since $K > 1$ and $2^{(r-1)/r} \geq 1$.

Now suppose that $\Phi \in C_\Phi$. If $g \in L_\infty$, then $g(t) \leq \|g\|_\infty$ and $g^*(t) \leq \|g\|_\infty$, so

$$\begin{aligned}
\|g\|_{M_{\Phi,1}} &= \sup_{x>0}\left(\frac{1}{\Phi(x)}\int_0^x g^*(t)\frac{\Phi(t)}{t}dt\right) \\
&\leq \|g\|_\infty \sup_{x>0}\frac{1}{\Phi(x)}\int_0^x \frac{\Phi(t)}{t}dt.
\end{aligned} \tag{B.3}$$

Using (4) in Definition 1, we get

$$\|g\|_{M_{\Phi,1}} \leq C\|g\|_\infty. \tag{B.4}$$

On the other hand, using (2) and (3) in Definition 1, we have

$$\begin{aligned}
\|g\|_{M_{\Phi,1}} &\geq \frac{1}{\Phi(x)}\int_0^x g^*(t)\frac{\Phi(t)}{t} \\
&\geq \frac{g^*(x)}{\Phi(x)}\int_0^x \frac{\Phi(t)}{t}dt \geq g^*(x).
\end{aligned} \tag{B.5}$$

So

$$\|g\|_{M_{\Phi,1}} \geq \lim_{x\to 0} g^*(x) = \|g\|_\infty. \tag{B.6}$$

The result then follows by combining inequalities (B.4) and (B.6).

C. Proof of Theorem 10

It easy to show that

$$\|g\|_{\Sigma_\Phi^1(\mu)} \cong \sup_{\mu(A)\neq 0}\frac{1}{\Phi(\mu(A))}\int_A |g(t)|\,d\mu(t). \tag{C.1}$$

Let A be μ-measurable subset of $[0,2\pi]$. For a μ-measurable set $A \in [0,2\pi]$, we have (see [7, exercise 1.4.5, page 65])

$$\int_A |g(t)|\,d\mu(t) = \int_0^{\mu(A)} g^*(s)\,ds. \tag{C.2}$$

Therefore using (C.1) and (C.2) we can show easily that

$$\|g\|_{\Sigma^1_\Phi(\mu)} \cong \sup_{\mu(A)\neq 0} \frac{1}{\Phi(\mu(A))} \int_A |g(t)|\, d\mu(t) \cong \sup_{t>0} \frac{1}{\Phi(t)} \int_0^t g^*(s)\, ds. \tag{C.3}$$

We only need to prove that $\|g\|_{L^\infty_\Psi} \cong \sup_{t>0}(1/\Phi(t)) \int_0^t g^*(s)ds$ to conclude.

Suppose that $g \in L^\infty_\Psi$ with $\Psi(t) = t/\Phi(t)$, $\Phi \in C_\Phi$. Then, for all $t > 0$, $(t/\Phi(t))g^*(t) \le \|g\|_{L^\infty_\Psi}$. Integrating both sides on the interval $[0, s]$, we have

$$\int_0^s g^*(s)\, dt \le \|g\|_{L^\infty_\Psi} \int_0^s \frac{\Phi(t)}{t} dt. \tag{C.4}$$

Using Dini's conditions above and taking the supremum over $s > 0$, we have

$$\sup_{s>0} \frac{1}{\Phi(s)} \int_0^s g^*(t)\, dt \le C\|g\|_{L^\infty_\Psi}. \tag{C.5}$$

On the other hand, since g^* is decreasing, for $s > 0$, we have

$$\frac{s}{\Phi(s)} g^*(s) \le \frac{1}{\Phi(s)} \int_0^s g^*(t)\, dt. \tag{C.6}$$

Taking the supremum over $s > 0$, we have

$$\|g\|_{L^\infty_\Psi} \le \sup_{t>0} \frac{1}{\Phi(t)} \int_0^t g^*(s)\, ds. \tag{C.7}$$

The equivalence follows by combining (C.5) and (C.7).

D. Proof of Theorem 11

First consider $f \in L_\Phi$ and $g \in L^\infty_\Psi$. Using a result by Hardy and Littlewood (see e.g., Exercise 1.4.1(b) in [7]), we have

$$\left|\int_0^{2\pi} f(t)\, g(t)\, dt\right| \le \int_0^{2\pi} f^*(t)\, g^*(t)\, dt \le \sup_{t>0}\left(\frac{t}{\Phi(t)} g^*(t)\right) \int_0^{2\pi} f^*(t) \frac{\Phi(t)}{t} dt \le \|g\|_{L^\infty_\Psi} \|f\|_{L_\Psi}. \tag{D.1}$$

For the second part, we start with $f(t) = \chi_A(t)$ for some μ-measurable subset A of $[0, 2\pi]$ and $g \in \Sigma^1_\Phi(\mu)$. Then

$$\int_0^{2\pi} f(t)\, g(t)\, d\mu(t) = \Phi(\mu(A)) \cdot \frac{1}{\Phi(\mu(A))} \int_A g(t)\, d\mu(t). \tag{D.2}$$

Thus

$$\left|\int_0^{2\pi} f(t)\, g(t)\, d\mu(t)\right| \le \Phi(\mu(A)) \sup_{\mu(A)\neq 0} \frac{1}{\Phi(\mu(A))} \left|\int_A g(t)\, d\mu(t)\right| = \Phi(\mu(A)) \|g\|_{\Sigma^1_\Phi(\mu)}. \tag{D.3}$$

So if $f(t) = \sum_{n=1}^\infty c_n \chi_{A_n}(t)$, the linearity of the integral gives us

$$\left|\int_0^{2\pi} f(t)\, g(t)\, d\mu(t)\right| \le \left(\sum_{n=1}^\infty |c_n|\, \Phi(\mu(A_n))\right) \|g\|_{\Sigma^1_\Phi(\mu)}. \tag{D.4}$$

Thus, taking the infimum over all the representations of f, we have

$$\left|\int_0^{2\pi} f(t)\, g(t)\, d\mu(t)\right| \le \|f\|_{A_\Phi(\mu)} \|g\|_{\Sigma^1_\Phi(\mu)}. \tag{D.5}$$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are grateful to the anonymous referees for their comments and suggestions that helped improve the quality of this paper.

References

[1] G. G. Lorentz, "Some new functional spaces," *Annals of Mathematics*, vol. 51, pp. 37–55, 1950.

[2] G. G. Lorentz, "On the theory of spaces Λ," *Pacific Journal of Mathematics*, vol. 1, pp. 411–429, 1951.

[3] M. J. Carro and J. Soria, "Weighted Lorentz spaces and the Hardy operator," *Journal of Functional Analysis*, vol. 112, no. 2, pp. 480–494, 1993.

[4] E. Kwessi, P. Alfonso, G. de Souza, and A. Abebe, "A note on multiplication and composition operators in Lorentz spaces," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 293613, 10 pages, 2012.

[5] R. R. Coifman, "A real variable characterization of H^p," *Studia Mathematica*, vol. 51, pp. 241–250, 1974.

[6] J. Jiao, L. Peng, and P. Liu, "Atomic decompositions of Lorentz martingale spaces and applications," *Journal of Function Spaces and Applications*, vol. 7, no. 2, pp. 153–166, 2009.

[7] L. Grafakos, *Classical Fourier Analysis*, Graduate Texts in Mathematics, Springer, 2nd edition, 2008.

[8] G. de Souza, "A new characterization of the Lorentz spaces $L(p, 1)$ for $P > 1$ and applications," in *Proceedings of the 2010 Real Analysis Exchange Summer Symposium*, pp. 55–58, 2010.

[9] G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*, Wiley-Interscience, 2nd edition, 1999.

[10] F. F. Bonsall, "Decompositions of functions as sums of elementary functions," *The Quarterly Journal of Mathematics. Second Series*, vol. 2, pp. 355–365, 1986.

[11] G. de Souza, "The atomic decomposition of Besov-Bergman-Lipschitz spaces," *Proceedings of the American Mathematical Society*, vol. 94, no. 4, pp. 682–686, 1985.

[12] G. de Souza, *Spaces formed by special atoms [Ph.D. dissertation]*, SUNY at Albany, 1980.

[13] S. Bloom and G. S. de Souza, "Atomic decomposition of generalized Lipschitz spaces," *Illinois Journal of Mathematics*, vol. 33, no. 2, pp. 181–209, 1989.

[14] G. S. de Souza and G. Sampson, "A real characterization of the pre-dual of Bloch functions," *The Journal of the London Mathematical Society*, vol. 27, no. 2, pp. 267–276, 1983.

[15] E. M. Stein and G. Weiss, "An extension of a theorem of Marcinkiewicz and some of its applications," *Journal of Applied Mathematics and Mechanics*, vol. 8, pp. 263–284, 1959.