Analytic characterization of high dimension weighted special atom spaces

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Abstract

Special atom spaces have been around for quite awhile since the introduction of atoms by R. Coifman in his seminal paper who led to another proof that the dual of the Hardy space H^1 is in fact the space of functions of bounded means oscillations (BMO). Special atom spaces enjoy quite a few attributes of their own, among which the fact that they have an analytic extension to the unit disc. Recently, an extension of special atom spaces to higher dimensions was proposed, making ripe the possible exploration of the above extension in higher dimensions. In this paper we propose an analytic characterization of special atom spaces in higher dimensions.

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1 Introduction

Let d be some positive integer. We define the unit disk as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the sphere as $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the polydisk and polysphere respectively as $\mathbb{D}^d$ and $\mathbb{T}^d$. Atoms were introduced by Coifman in [5] as a tool to explicitly represent functions in the Hardy space H^p for $0 < p \leq 1$. The following definition was proposed:

Definition 1.1. *Let $0 < p \leq 1$ and an interval J of $\mathbb{R}$. An atom is a function b defined on the interval J and satisfying*

1. $|b(\xi)| \leq \dfrac{1}{|J|^{1/p}}$.
2. $\displaystyle\int_{-\infty}^{\infty} \xi^k b(\xi) d\xi = 0,$ *for* $0 \leq k \leq \left[\frac{1}{p}\right] - 1$, *where* $[x]$ *is the integer part of* x .

From this definition, functions in $H^p(\mathbb{R})$ could now be characterized via their atomic decomposition in the following theorem:

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Theorem 1.2 (see [5]). *Let $0 < p \leq 1$. Then $f \in H^p(\mathbb{R})$ if and only if there exist real numbers a_i, atoms b_i for $i \in \mathbb{N}$, and absolute constants c, C such that such that*

$$f(\xi) = \sum_{i=0}^{\infty} a_i b_i(\xi) \quad \text{and} \quad c \|f\|_{H^p} \leq \sum_{i=0}^{\infty} |a_i|^p \leq C \|f\|_{H^p} \ .$$

Fefferman [8] observed that this result is in fact due to the duality between $H^1(\mathbb{R})$ and the space of functions of bounded means oscillations (BMO), therefore providing another proof that the dual space of $H^1(\mathbb{R})$ is in fact BMO. The era of the atomic decomposition therefore started. One criticism of the atomic decomposition at the time was that it was too general, making it difficult or not very useful for applications. This atomic decomposition was proved to be quite useful in harmonic analysis. However, in an attempt to answer this criticism, Richard O'Neil and Geraldo De Souza proposed an example of atoms defined on the interval $I = [0, 1]$ that was latter dubbed "special atoms". This special atom has some very desirable properties as we will see in the sequel.

Definition 1.3. *Consider $1 \leq p < \infty$.*

(a) A **special atom of type 1** *is a function $b : I \to \mathbb{R}$ such that*

$$b(\xi) = \begin{cases} \dfrac{1}{|J|^{1/p}} [\chi_R(\xi) - \chi_L(\xi)], & \text{if } \xi \in J \\ 1, & \text{if } \xi \in I \backslash J \end{cases} ,$$

where J is a subinterval of I, L and R are the halves of J such that $J = L \cup R$, and $|J|$ is the length of J .

(b) A **special atom of type 2** *is a function $c : J \to \mathbb{R}$ such that*

$$c(\xi) = \frac{1}{|J|^{1/p}} [\chi_J(\xi)] \ ,$$

where J is an interval contained in I .

Remark 1.4. *We observe that this definition can be extended on the unit ball of $\mathbb{R}^d$ by using dyadic decomposition, see [1] .*

From this definition, they introduced the *special atom space* B^p (for type 1 atom) defined on J, but with a different norm from the L^p-norm.

Definition 1.5. *Let $1 \leq p < \infty$. The special atom space B^p (of type 1) is defined as*

$$B^p = \left\{ f : I \to \mathbb{R}; f(\xi) = \sum_{n=0}^{\infty} \alpha_n b_n(\xi); \sum_{n=0}^{\infty} |\alpha_n| < \infty \right\} ,$$

where the b_n's are special atoms of type 1. The space B^p is endowed with the norm

$$\|f\|_{B^p} = \inf \sum_{n=0}^{\infty} |\alpha_n| \, ,$$

where the infimum is taken over all representations of f .

The weighted special atom soon followed (see for instance [3]) which gave rise a host of very interesting properties, namely the analytic characterization.

Definition 1.6. *We define the* **weighted special atom (of type 1)** *on J as:*

$$b_w(\xi) = \frac{1}{w(J)} \left[\chi_{_R}(\xi) - \chi_{_L}(\xi)\right] ,$$

where

$$w \in L^1(I) \quad \text{with} \quad w(J) = \int_J w(\xi) d\xi, \quad \text{and } L, R \text{ are as in Definition 1.3}\,.$$

The weighted special atom space is the space B_w of functions f with atomic decomposition

$$f(\xi) = \sum_{n=0}^{\infty} \alpha_n b_{w,n}(\xi) ,$$

endowed with the Infimum norm.

The importance of this definition can not be overstated. Indeed, weighted special atom spaces are invariant under the Hilbert transform and they contain some functions whose Fourier series diverge, see [7]. One of their most applicable features is their connection to Haar wavelets, in that, a Haar wavelet function is just a special atom with weight $2^{-n/2}$, [13]. Weighted special atom spaces are also Banach equivalent to some Bergman-Besov-Lipschitz spaces (see [6]), which leads to a complete characterization of their lacunary functions, see [12]. Moreover, functions in B_w have analytic correspondences by integrating against analytic functions whose real parts coincide with the Poisson kernel. In particular, B^1 is Banach equivalent to the space of analytic functions F on the complex unit disc for which $F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\xi} + z}{e^{i\xi} - z} f(\xi) d\xi$. The question that was latter raised by Brett Wick in 2010 (personal communication with the first author) was whether this analytic characterization could be achieved in higher dimensions. In order to entertain such a question, one has to, for $d \geq 2$,

1. first provide a definition of the special atom and its weighted counterpart on I^d so that its restriction to $I = [0, 1]$ is the original special atom.

2. second, provide a definition of the special atom space B^p on I^d.

3. third, verify that the Banach structure of B^p is preserved.

4. fourth, set the conditions on the weight function w on I^d and define the weighted special atom space B_w on I^d.

5. fifth, define the analytic extension $F(\boldsymbol{z})$ of a function $f(\boldsymbol{\xi}) \in B_w$ for $\boldsymbol{z} = (z_1, z_2, \cdots, z_d) \in \mathbb{D}^d$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_d) \in I^d$.

6. sixth, verify that B_w and its analytic extension A_w^1 are indeed Banach-equivalent under the above conditions.

Remark 1.7. *The first step was recently accomplished in [13]. Also the requirement that by restricting to $I = [0, 1]$ we obtain the original special atom is for simplicity sake. The argument is important in high dimensions to prove for example in the case of Haar wavelets that we obtain an orthonormal system. However, there exist numerous ways to define atoms similar to the special atom.*

We end this introductory part by recalling the definition of the weighted Lipschitz class of functions.

Definition 1.8. *Let w be a weight function defined on $J = [a - h, a + h] \subseteq I$. The weighted Lipschitz class is the class of continuous functions defined as*

$$\Lambda_w = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{\Lambda_w} = \sup_{h>0,\xi} \left| \frac{f(\xi + h) + f(\xi - h) - 2f(\xi)}{w(J)} \right| < \infty. \right\}$$

For completeness, recall that for $w(t) = t$, Λ_w is the Zygmund class and for $w(t) = t^\alpha, 0 < \alpha < 2$, Λ_w is the Lipschitz class of order α. It was proved in [3] that the dual space B_w^* of B_w is $\Lambda_w' = \{f' : f \in \Lambda_w\}$, where f' is understood in the sense of distributions.
The remainder of the paper is organized as follows: In Section 2, we show how to extend weighted special atoms to high dimensions. In the last step, we state the Main Theorem in Section 3, and we will make concluding remarks in Section 4.

2 High Dimension Extension

Let $\boldsymbol{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ for an integer $d \geq 1$. In fact, in the sequel, bold-faced symbols will represent vectors. We start out by proposing a definition of a weighted special atom in higher dimensions, for a general weight function w. When w is the Lebesgue measure, the interested reader can refer to [13] for a more constructive approach in the definition.

Definition 2.1. *Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$ and $1 \leq p < \infty$.*

(a) Let $J := \prod_{j=1}^{d} [a_j - h_j, a_j + h_j]$ where a_j, h_j are real numbers with $h_j > 0$. The weighted special atom (of type 1) on J, a sub-interval of I^d, is defined as

$$b_w(\boldsymbol{\xi}) = \frac{1}{w(J)} \left\{ \chi_R(\boldsymbol{\xi}) - \chi_L(\boldsymbol{\xi}) \right\},$$

where $w(J) = \int_J w(\boldsymbol{\xi}) d\boldsymbol{\xi}$ and $R = \bigcup_{j=1}^{2^{d-1}} J_{i_j}$ for some $(i_1, i_2, \cdots, i_{2^{d-1}} \in \left\{1, 1, \cdots, 2^d\right\})$ with $i_1 < i_2 < \cdots < i_{2^{d-1}}$ and $L = J \backslash R$. $\{J_1, J_2, \cdots, J_{2^d}\}$ is the collection of sub-cubes of J, cut by the hyperplanes $x_1 = a_1, x_2, \cdots, x_d = a_d$, and χ_A represents the characteristic function of set A.

(b) The weighted special atom space B_w is the space of real-valued functions f defined on I^d such that

$$f(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \alpha_n b_{w,n}(\boldsymbol{\xi}) \quad \textit{with} \sum_{n=0}^{\infty} |\alpha_n| < \infty,$$

endowed with the norm

$$\|f\|_{B_w} = \inf \sum_{n=0}^{\infty} |\alpha_n| \ ,$$

where the infimum is taken over all possible representations of f.

For example, for $d = 2$, for real numbers a_1, a_2, h_1, and h_2 such that $h_1, h_2 > 0$, consider a sub-interval J of I^d defined as $J = [a_1 - h_1, a_1 + h_1] \times [a_2 - h_2, a_2 + h_2]$. Let

$$\begin{aligned} L_1 &= [a_1 - h_1, a_1] \times [a_2 - h_2, a_2], \quad L_2 = [a_1 - h_1, a_1) \times [a_2, a_2 + h_2], \\ R_1 &= [a_1, a_1 + h_1] \times [a_2 - h_2, a_2), \quad R_2 = (a_1, a_1 + h_1] \times (a_2, a_2 + h_2] \ . \end{aligned}$$

Consider

$$L = L_1 \cup R_2 \quad \text{and} \quad R = L_2 \cup R_1 \ .$$

The special atom $b(\xi_1, \xi_2)$ is then defined as:

$$\begin{aligned} b_w(\xi_1, \xi_2) &= \frac{1}{w(J)} \Big\{ \chi_{_R}(\xi_1, \xi_2) - \chi_{_L}(\xi_1, \xi_2) \Big\} \\ &= \frac{1}{w(J)} \Big\{ \chi_{_{L_2}}(\xi_1, \xi_2) + \chi_{_{R_1}}(\xi_1, \xi_2) - \chi_{_{L_1}}(\xi_1, \xi_2) - \chi_{_{R_2}}(\xi_1, \xi_2) \Big\} \ . \end{aligned}$$

For $d \geq 2$, we consider $J = \prod_{j=1}^{d} [a_j - h_j, a_j + h_j]$ where a_j, h_j are real numbers with $h_j > 0$. For $j = 1, \cdots, d$, we define $b_w(\boldsymbol{\xi})$ similarly. Figure 1 below is an illustration of b_w for $d = 2$ (a) and $d = 3$ (b) when w is the Lebesgue measure.

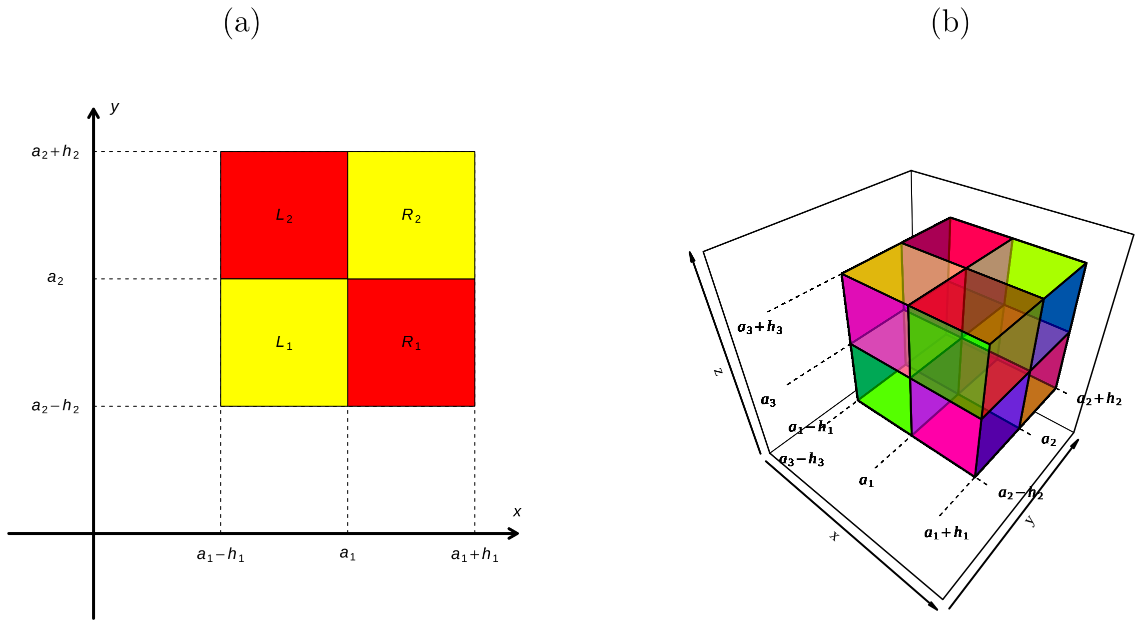

Figure 1: (a) represents a special Atom for $d = 2$ and (b) represents the special atom for $d = 3$. Note that for $d = 3$, the color areas represent the different partitions of J_j into subintervals R and L intervals.

With this definition, we can prove the following theorem about the Banach structure of B_w.

Theorem 2.2. *For $1 \le p < \infty$, $\left(B_w, \|\cdot\|_{B_w}\right)$ is a Banach space.*

Proof. The proof can be seen in Section 3 below. □

Definition 2.3.
Consider the function $P : \mathbb{D}^d \times I^d$ defined as

$$P(\boldsymbol{z}, \boldsymbol{\xi}) = \prod_{j=1}^{d} P_j(z_j, \xi_j) \quad \textit{where} \quad P_j(z_j, \xi_j) = \frac{e^{i\xi_j} + z_j}{e^{i\xi_j} - z_j}\ .$$

We observe that for fixed $1 \le j \le d$ and $z_j \in \mathbb{D}$, $\mathrm{Re}(P_j(z_j, \xi_j))$ is the Poisson Kernel.
For a function F defined on $\mathbb{D}^d$, we define $F'(\boldsymbol{z})$ as

$$F'(\boldsymbol{z}) = (f_1(\boldsymbol{z}), \cdots, f_d(\boldsymbol{z})), \quad \text{where} \quad f_j(\boldsymbol{z}) = \frac{\partial F(\boldsymbol{z})}{\partial z_j}\ .$$

In the sequel $d\boldsymbol{\xi} = d\xi_1 d\xi_2 \cdots d\xi_d$, and this will be the case for similar bold-faced symbols.
Also for a set $A \subseteq S$, we will denote by $A^c = S \backslash A$.
Now we give the definition of special weights functions that will be necessary for the Proof of the main theorem

Definition 2.4. *Let w be a real-valued function defined on $[0,1]$. Let m and n be positive integers.*

(a) Then w is said to be **Dini** *of order $m \geq 1$ and we denote $w \in \mathcal{D}_m$ if $\dfrac{w(u)}{u^m} \in L^1(0,1)$, and there exists an absolute constant C for which*

$$\int_0^u \frac{w(\xi)}{\xi^m} d\xi \leq C w(u), \quad \text{for } 0 < u < 1 .$$

(b) A function $w : [0,\infty) \to \mathbb{R}$ is said to be in the class $\mathcal{B}_n$ for some positive integer n and we denote $w \in \mathcal{B}_n$ if for $0 < u < 1$,

1. *w is increasing and $w(0) = 0$.*
2. *There exists a constant C such that $\displaystyle\int_u^1 \frac{w(\xi)}{\xi^{n+1}} d\xi \leq C \frac{w(u)}{u^n}$.*

(c) Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_d) \in \mathbb{R}^d$. For $1 \leq j \leq d$, consider weight functions w_j defined on $\mathbb{R}_+$. We define the product weight function $w(\boldsymbol{\xi})$ as

$$w(\boldsymbol{\xi}) = \prod_{j=1}^{d} w_j(\xi_j), \tag{2.1}$$

(d) A weight w is said to be in the class $\mathscr{B}_p$ on $[0,1]$ if there exists a constant C such that for any interval $J \subseteq [0,1]$ with center ξ_J, we have

$$\frac{|J|^p}{w(J)} \int_{J^c} \frac{w(\xi)}{|\xi - \xi_J|^p} d\xi \leq C .$$

Remark 2.5.

1. *We observe that $w \in \mathscr{B}_p$ for some $p > 1$ if and only if w is a doubling measure, that is, there exists an absolute constant C such that*

$$w[Q_{2h}(\xi)] \leq C w[Q_h(\xi)], \quad \text{where } Q_h(\xi) = \{t \in \mathbb{R} : |\xi - t| \leq h\} .$$

The class of doubling weights will be referred to as $\mathscr{D}$.

2. *The Muckenhoupt class of weights, (see [14])*

$$\mathscr{A}_p = \left\{ w \in L^1 : \left(\frac{1}{|I|} \int_I w(\xi) d\xi \right) \left(\frac{1}{|I|} \int_I w^{1/1-p}(\xi) d\xi \right)^{p-1} < \infty \right\}$$

is strictly contained in the class $\mathscr{B}_p$.

The following is an important lemma relating class $\mathcal{D}_m, \mathcal{B}_n$ and $\mathscr{D}$.

Lemma 2.6. *Let w be a weight function. Then for all integer $n \geq 1$, we have:*

$$\mathcal{D}_1 \cap \mathcal{B}_n \subseteq \mathscr{D} .$$

Proof. Fix $n \geq 1$ and let $w \in \mathcal{D}_1 \cap \mathcal{B}_n$. We would like to show that for u such that $2u \leq 1$ there exists an absolute constant C such that

$$\int_0^{2u} \frac{w(\xi)}{\xi} d\xi \leq C w(u) \ .$$

We have that

$$\int_0^{2u} \frac{w(\xi)}{\xi} d\xi = \int_0^{u} \frac{w(\xi)}{\xi} d\xi + \int_u^{2u} \frac{w(\xi)}{\xi} d\xi = I_1 + I_2 \ .$$

On one hand, since $w \in \mathcal{D}_1$, there exists $C > 0$ such that $I_1 \leq C \cdot w(u)$. On the other hand,

$$\begin{aligned} I_2 = \int_u^{2u} \frac{w(\xi)}{\xi} d\xi &= u^n \int_u^{2u} \frac{w(\xi)}{\xi u^n} d\xi \\ &\leq u^n 2^{n+1} \int_u^{2u} \frac{w(\xi)}{\xi^{n+1}} d\xi, \quad \text{since } \tfrac{1}{\xi u^n} \leq \tfrac{1}{u^{n+1}} \leq \tfrac{2^{n+1}}{\xi^{n+1}} \\ &\leq C \cdot 2^{n+1} \cdot w(u), \quad \text{since } w \in \mathcal{B}_n \\ &\leq C \cdot w(u) \ . \end{aligned}$$

It follows that $I_1 + I_2 \leq C \cdot w(t)$ and thus $w \in \mathscr{D}$. □

Definition 2.7. *A product weight $w \in \mathcal{D}_m$ (respectively $w \in \mathcal{B}_n$ or $w \in \mathscr{B}_p$) if for each $1 \leq j \leq d$, the weight $w_j \in \mathcal{D}_m$ (respectively $w_j \in \mathcal{B}_n$ or $w_j \in \mathscr{B}_p$) for integers $m, n \geq 1$ and a real number $p > 1$.*

We now introduce spaces of analytic functions that will be proven to be the analytic characterizations of weighted special atom spaces.

Definition 2.8. *Let $1 \leq p < \infty$ be a real number. Given a function $w : \mathbb{D}^d \to \mathbb{R}_+$, we will consider A^p_w as the space of analytic functions F defined on $\mathbb{D}^d$ such that*

$$\|F\|_{A^p_w} = |F(\mathbf{0})| + \frac{1}{(2\pi)^d} \int_{\mathbb{D}^d} \left|F'(\boldsymbol{r} e^{i\boldsymbol{\xi}})\right|^p w(\boldsymbol{r} e^{i\boldsymbol{\xi}}) d\boldsymbol{\xi} d\boldsymbol{r} < \infty, \tag{2.2}$$

where

$$\left|F'(\boldsymbol{r} e^{i\boldsymbol{\xi}})\right| = \left\|F'(\boldsymbol{r} e^{i\boldsymbol{\xi}})\right\|_2 = \sqrt{|f_1|^2 + \cdots + |f_d|^2} \quad \text{and } \boldsymbol{z} = \boldsymbol{r} e^{i\boldsymbol{\xi}} \ . \tag{2.3}$$

Remark 2.9.

1. *We note that for $d = 1$ and $w(re^{i\xi}) = 1$, A^1_w is contained in the Hardy space $H^1(\mathbb{D}^d)$.*

2. *By Holder's inequality, $A^p_w \subseteq A^1_w$ for $p > 1$. In particular, A^1_w contains the weighted Dirichlet space D_w (see [11]) of analytic functions F such that*

$$\int_{\mathbb{D}} \left|F'(re^{i\xi})\right|^2 w(re^{i\xi}) dr d\xi < \infty \ .$$

3. *When* $r = |z|^2$ *and* $w(re^{i\xi}) = (1-r)^{\alpha} = (1-|z|^2)^{\alpha}$ *for* $\alpha > 0$, *then* A^1_w *contains the Bloch space* B_α *(see [16]) of analytic functions* F *such that*

$$\sup_{\mathbb{D}} \left\{(1-|z|^2)^{\alpha} |F'(z)|\right\} < \infty .$$

Definition 2.10. *Let* $f \in B_w$. *We define the analytic extension* F *of* f *as*

$$F(\boldsymbol{z}) = \frac{1}{(2\pi)^d} \int_I P(\boldsymbol{z}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} , \tag{2.4}$$

in the sense that we can recover f *by taking the radial limit*

$$f(\boldsymbol{\xi}) = \lim_{\boldsymbol{r}\to 1} Re\, F(\boldsymbol{r}e^{i\boldsymbol{\xi}}) ,$$

where the limit is taken element-wise. The space A^1_w *defined in equation (2.2) for some* $f \in B_w$ *will be referred to as the analytic extension of* B_w.

Remark 2.11.

1. *We observe that in the definition of the analytic extension* F *in (2.4), the function* $\log(F(z))$ *can be viewed as the outer function whereas* $f(\xi)$ *can be viewed as the inner function, similarly to a Beurling factorization, see for example Definition 17.14 in [15].*

2. *Additionally, in this definition, if one choose* $f(\boldsymbol{\xi}) = \log(1-|b(\boldsymbol{\xi})|^2)^{\frac{1}{2}}$, *then we can define the function* $a(z)$, *the so called-Pythagorean mate of* $b \in \mathcal{H}(b)$ *(The interested reader can refer for example to [10] for more on these spaces) as*

$$a(\boldsymbol{z}) = exp\left(\int_I \frac{\boldsymbol{\xi}+\boldsymbol{z}}{\boldsymbol{\xi}-\boldsymbol{z}} \log(1-|b(\boldsymbol{\xi})|^2)^{\frac{1}{2}} d\boldsymbol{\xi}\right)$$

   This suggests that the space A^p_w *is closely related to the theory of* $\mathcal{H}(b)$ *spaces, but more importantly, can be used as a gateway to their study in higher dimensions. We know that there is an extensive literature on these spaces* $\mathcal{H}(b)$ *in one dimension, see [2, 10, 4].*

3. *Moreover, if* $f \in L^p(\mathbb{T}^d)$ *for* $1 < p < \infty$, *then by Theorem 17.26 in [15], the analytic extension* $F \in H(\mathbb{D}^d)$, *the Hardy's space on the polydisk.*

We will prove in the sequel that when the weight function w satisfies certain conditions, then the spaces B_w and A^1_w are in fact isometric to each other, that is, the inclusion operator $G : B_w \to A^1_w, G(f) = F$ is a Banach isometry.

3 Main results

Now we can now state our main theorem:

Main Theorem. *Let* $J_j = [a_j - h_j, a_j + h_j]$, $J = \prod_{j=1}^{d} J_j$. *Let* w *be a weight defined on* J. *Let* A_w *be the space of analytic functions defined above. Then we have the following:*

(a) $B_w \subseteq A_w^1$ *if and only if* $w(\boldsymbol{r}e^{i\boldsymbol{\xi}}) \equiv w(\boldsymbol{\xi})$ *is a product weight and* $w \in \mathscr{B}_2$.

(b) B_w *is Banach equivalent to* A_w^1 *if and only if* $w(\boldsymbol{r}e^{i\boldsymbol{\xi}}) \equiv \dfrac{w(1-\boldsymbol{r})}{1-\boldsymbol{r}}$ *is a product weight and* $w \in \mathcal{D}_1 \cap \mathcal{B}_2$.

Remark 3.1.
The Main theorem essentially states that B_w *and* A_w^1 *are isomorphic as Banach spaces in the sense that*

1. B_w *and* A_w^1 *are both Banach spaces,*
2. $f \in B_w$ *if and only if its analytic extension* $F \in A_w^1$,
3. $F \in A_w^1$ *if and only if* $\lim_{\boldsymbol{r}\to 1} \mathrm{Re}\, F(\boldsymbol{r}e^{i\boldsymbol{\xi}}) \in B_w$,
4. $\|f\|_{B_w} \equiv \|F\|_{A_w^1}$.

Remark 3.2.

1. *In the first part of the Main Theorem, the weight function depends only on the argument* $\boldsymbol{\xi}$ *of* $\boldsymbol{z} = \boldsymbol{r}e^{i\boldsymbol{\xi}}$, *whereas in the second part, it depends only on the radius* $\boldsymbol{r}$. *The condition that* $w \in \mathscr{B}_2$ *in the first part is weaker than the condition* $w \in \mathcal{D}_1 \cap \mathcal{B}_2$ *in the second part since per Lemma 2.6,* $w \in \mathcal{D}_1 \cap \mathcal{B}_2 \subseteq \mathscr{D} = \bigcup_{p>1} \mathscr{B}_p$ *implies the existence of* $p > 1$ *such that* $w \in \mathscr{B}_p$. *There is no guarantee that this* p *will be 2 as in the first part. However, what the two parts have in common is the necessary condition that if* B_w *is contained in* A_w^1, *then the weight* $w \in \mathscr{D}$.
2. *In the Main Theorem, the weight* w *is a product weight, however general weights* w *defined on* I^d *are not addressed in this manuscript and it would be a worthwhile future endeavor to have a holistic understanding of the role of the weight* w.

The proof of the Main Theorem relies on some crucial lemmas that will be stated below. The first lemma shows that partial derivatives of analytic extensions of special atoms in higher dimensions are bounded. Henceforth, the constants C will be generic and when necessary, their dependence on an interval J will be specified accordingly.

Lemma 3.3. *Let* $J_j = [a_j - h_j, a_j + h_j]$, $J = \prod_{j=1}^{d} J_j$ *and* $F(z) = \dfrac{1}{(2\pi)^d} \displaystyle\int_J P(\boldsymbol{z}, \boldsymbol{\xi}) b_w(\boldsymbol{\xi}) d\boldsymbol{\xi}$. *Then for any* $j = 1, \cdots, d$, *(1) there exists a constant $C(J)$ such that*

$$f_j(z_j) = C(J)K_1(a_j,h_j,z_j)\prod_{\substack{l=1\\ l\neq j}}^{d} K_2(a_l,h_l,z_l)\ , \tag{3.1}$$

where

$$\begin{aligned} K_1(a_j,h_j,z_j) &= \frac{1}{i}\left[\frac{1}{z_j-e^{i(a_j-h_j)}}+\frac{1}{z_j-e^{i(a_j+h_j)}}+\frac{2}{e^{ia_j}-z_j}\right]\ , \\ K_2(a_l,h_l,z_l) &= \frac{2}{i}\left[\ln\left(e^{i(a_l-h_l)}-z_l\right)+\ln\left(e^{i(a_l+h_l)}-z_l\right)-2\ln\left(e^{ia_l}-z_l\right)\right]\ . \end{aligned}$$

(2) Moreover for $i,j=1,\cdots,k$, there are absolute constants C_1 and C_2 such that

$$|K_1(a_j,h_j,z_j)|\le C_1,\quad |K_2(a_l,h_l,z_l)|\le C_2\ .$$

Lemma 3.4. *Let real numbers a and $h>0$, and $z\in\mathbb{D}$. Let $J=[a-h,a+h]$.*

(a) If $w\in\mathscr{B}_2$ such that $w(re^{i\xi})\equiv w(\xi)$, then there exists a constant C such that

$$\int\int_{\mathbb{D}} |K_1(a,h,z)|\,w(\xi)d\xi dr\le C(J)<\infty\ .$$

(b) If $w\in\mathcal{D}_1\cap\mathcal{B}_2$ such that $w(re^{i\xi})\equiv\dfrac{w(1-r)}{1-r}$, then there exists a constant C such that

$$\int\int_{\mathbb{D}} |K_1(a,h,z)|\,\frac{w(1-r)}{1-r}d\xi dr\le C(J)<\infty\ .$$

Lemma 3.5. *Let $1\le j\le d$ and $J_j=[a_j-h_j,a_j+h_j]$ for real numbers a_j and $h_j>0$. Consider $z_j=r_je^{i\xi_j}\in\mathbb{D}$ such that $h_j<|e^{ia_j}-z_j|$. Consider a product weight w such that $\frac{w_j(t)}{t^2}\in L^1(0,1)$ for all $1\le j\le d$. Then there exists a constant $C(J_j)>0$ such that*

$$\frac{h_j^2}{w_j(J_j)}\int_{\xi_j\notin J}\frac{\omega(\xi_j)}{\xi_j^2}d\xi_j\le C(J_j)\int\int_{\mathbb{D}}|f_j(\boldsymbol{z})|\,d\xi_j dr_j.$$

Lemma 3.6. *Let $j=[a-h,a+h]$ for real numbers a and $h>0$ and $w(t)/t$ and in $L^1(J)$. Fix $1\le j\le d$.*

(a) Consider $D_1=\left\{z=re^{i\xi}\in\mathbb{D}: h<|e^{ia}-z|\right\}$. There exists an absolute constant C such that

$$\int\int_{D_1}|f_j(\boldsymbol{z})|\,\frac{w(1-r)}{1-r}d\xi dr\ge C_j\int_h^1\frac{w(u)}{u}du$$

(b) Consider the subset $D_2=\left\{z\in\mathbb{D}:|e^{ia}-z|\le\frac{h}{4}\right\}$. There exists an absolute constant C_j such that

$$\int\int_{D_2}|f_j(\boldsymbol{z})|\,\frac{w_j(1-r)}{1-r}d\xi dr\ge C_j\int_0^{\frac{h}{4\sqrt{2}}}\frac{w(u)}{u}du\ .$$

Proof of the Main Theorem.

Part (a): Let w be a product weight such that $w \in \mathscr{B}_2$. To show that $B_w \subseteq A^1_w$, it will be enough to show that analytic extensions of weighted special atoms $b_w(\boldsymbol{\xi})$ are contained in A^1_w, that is, we will show that for a special atom $b_w(\boldsymbol{\xi}) \in B_w$, we have $F \in A^1_w$ where

$$F(\boldsymbol{z}) = \frac{1}{(2\pi)^d} \int_I P(\boldsymbol{z}, \boldsymbol{\xi}) b_w(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Fix $1 \le j \le d$. Let $d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j} = d\xi_1 dr_1 \cdots d\xi_{j-1} dr_{j-1} d\xi_{j+1} dr_{j+1} \cdots d\xi_d dr_d$. From Lemma 3.3 above, we have

$$\begin{aligned} |f_j(\boldsymbol{z})| &= C(J) \, |K_1(a_j, h_j, z_j)| \prod_{\substack{l=1 \\ l \ne j}}^{d} |K_2(a_l, h_l, z_l)| \\ &\le M(J) \, |K_1(a_j, h_j, z_j)| \quad \text{where} \quad M(J) = C(J) C_2^{d-1} \, . \end{aligned}$$

Also from Lemma 3.3 above, $K_1(a_j, h_j, z_j)$ is bounded, for all $1 \le j \le d$. Therefore $\sup\limits_{1 \le j \le d} |K_1(a_j, h_j, z_j)|$ exists. Moreover using the definition of $|F'(\boldsymbol{z})|$ in equation (2.3), we obtain

$$|F'(\boldsymbol{z})| \le d^{1/2} M(J) \sup_{1 \le j \le d} |K_1(a_j, h_j, z_j)| \, . \tag{3.2}$$

We then have that

$$\begin{aligned} \int\int_{\mathbb{D}^d} |F'(\boldsymbol{z})| \, w(\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{r} &= \int\int_{\mathbb{D}^d} |F'(\boldsymbol{z})| \left(\prod_{j=1}^{d} w_j(\xi_j) \right) d\boldsymbol{\xi} d\boldsymbol{r} \\ &\le d^{1/2} M(J) \left(\sup_{1 \le j \le d} \int\int_{\mathbb{D}} |K_1(a_j, h_j, z_j)| \, w_j(\xi_j) d\xi_j dr_j \right) \\ &\times \left(\int\int_{\mathbb{D}^{d-1}} \left(\prod_{\substack{l=1 \\ l \ne j}}^{d} w_l(\xi_l) \right) d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j} \right) \\ &\le C \sup_{1 \le j \le d} \left(\int\int_{\mathbb{D}} |K_1(a_j, h_j, z_j)| \, w_j(\xi_j) d\xi_j dr_j \right) < \infty \, . \end{aligned}$$

This proves that $F \in A^1_w$.

Conversely, suppose that $F \in A^1_w$. We will show that in this case, $w \in \mathscr{B}_2$. As above, it suffices to consider analytic extensions F of weighted special atoms. Let $C > 0$ such that $\|F\|_{A^1_w} < C$.

Fix $1 \le j \le d$. Then we know that $\|F'(\boldsymbol{z})\|_2 \ge \|F'(\boldsymbol{z})\|_\infty \ge |f_j(\boldsymbol{z})|$. Therefore

$$\begin{aligned} C > \int\int_{\mathbb{D}^d} |F'(\boldsymbol{z})|\, w(\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{r} &= \int\int_{\mathbb{D}^d} \left(\sum_{l=1}^{d} |f_l(\boldsymbol{z})|^2\right)^{1/2} w(\boldsymbol{\xi}) d\boldsymbol{\xi} d\boldsymbol{r} \\ &\ge \left(\int\int_{\mathbb{D}^{d-1}} \left(\prod_{\substack{l=1\\ l\ne j}}^{d} w_l(\xi_l)\right) d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j}\right) \\ &\times \left(\int\int_{\mathbb{D}} |f_j(\boldsymbol{z})|\, w_j(\xi_j) d\xi_j dr_j\right) \\ &\ge \left(\prod_{\substack{l=1\\ l\ne j}}^{d} \|w_l\|_{L^1}\right) \times \left(\int_{\xi_j \notin J_j} \frac{w_j(\xi_j)}{\xi_j^2} d\xi_j\right), \quad \text{by Lemma 3.5} \end{aligned}$$

Since j is arbitrary, it follows that

$$\frac{|J_j|^p}{w_j(J_j)} \int\int_{\xi_j \notin J_j} \frac{w_j(\xi_j)}{\xi_j^2} d\xi_j dr_j < C, \quad \forall j = 1, \cdots, d.$$

Therefore $w \in \mathscr{B}_2$ and this concludes the proof of part (a) of the theorem.
Part (b). Now assume w is a product weight such that $w \in \mathcal{D}_1 \cap \mathcal{B}_2$. Let $F \in B_w$. As above, we will proceed by showing that analytic extensions F of special atoms are in A^1_w. Thus consider $F(z) = \dfrac{1}{(2\pi)^d} \displaystyle\int_J P(\boldsymbol{z}, \boldsymbol{\xi}) b_w(\boldsymbol{\xi}) d\boldsymbol{\xi}$. We know from above equation (3.2) that

$$\begin{aligned} \int\int_{\mathbb{D}^d} |F'(\boldsymbol{z})| \frac{w(1-\boldsymbol{r})}{1-\boldsymbol{r}} d\boldsymbol{\xi} d\boldsymbol{r} \le d^{1/2} M(J) &\left(\sup_{1\le j\le d} \int\int_{\mathbb{D}} |K_1(a_j, h_j, z_j)| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j\right) \\ &\times \left(\int\int_{\mathbb{D}^{d-1}} \left(\prod_{\substack{l=1\\ l\ne j}}^{d} \frac{w_l(1-r_l)}{1-r_l}\right) d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j}\right) \end{aligned}$$

By Lemma 3.4, we have

$$\sup_{1\le j\le d} \int\int_{\mathbb{D}} |K_1(a_j, h_j, z_j)| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \le \sup_{1\le j\le d} C(J_j) < \infty\,.$$

By the Dini condition, we have that

$$C_0 = \int\int_{\mathbb{D}^{d-1}} \left(\prod_{\substack{l=1\\ l\ne j}}^{d} \frac{w_l(1-r_l)}{1-r_l}\right) d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j} = (2\pi)^{d-1} \prod_{\substack{l=1\\ l\ne j}}^{d} \left(\int_0^1 \frac{w_l(u_l)}{u_l} du_l\right) < \infty\,.$$

Therefore, we can infer that $F \in A^1_w$.
Conversely, suppose that $F \in A^1_w$. We will show that $w \in \mathcal{D}_1 \cap \mathcal{B}_2$. Let $C > 0$ such that

$\|F\|_{A^1_w} \le C$. Fix $1 \le j \le d$. Then as above,

$$C > \int\int_{\mathbb{D}^d} |F'(\boldsymbol{z})| \frac{w(1-\boldsymbol{r})}{1-\boldsymbol{r}} d\boldsymbol{\xi} d\boldsymbol{r} \ge \left(\int\int_{\mathbb{D}^{d-1}} \left(\prod_{\substack{l=1 \\ l \ne j}}^{d} \frac{w_l(1-r_l)}{1-r_l} \right) d\boldsymbol{\xi}^{-j} d\boldsymbol{r}^{-j} \right)$$
$$\times \left(\int\int_{\mathbb{D}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \right) \quad (3.3)$$
$$= C_0 \times \int\int_{\mathbb{D}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \,.$$

We can first combine the latter with equation (3.3) and Lemma 3.6 (a) to obtain that $\int_{h_j}^1 \frac{w_j(u_j)}{u_j^3} dr_j \le C_j$, that is, $w_j \in \mathcal{B}_2$. Since j is arbitrary, it follows that $w \in \mathcal{B}_2$. We can also combine the latter with (3.3), Lemma 2.6, and Lemma 3.6 (b) to conclude that $w_j \in \mathcal{D}_1$. Since j is arbitrary, it follows that $w \in \mathcal{D}_1$.
It remains to show that B_w and A^1_w are norm-equivalent. Since $B_w \subseteq A^1_w$, there exists a constant $M > 0$ such that $\|f\|_{A^1_w} \le M \|f\|_{B_w}$. To obtain the reverse inequality, it suffices to use a simple extension of the one dimension case to obtain that the dual B^*_w of B_w is continuously contained in the dual A^{1*}_w of A^1_w. Hence by virtue of the inclusion $B_w \subseteq A^1_w$, we have $A^{1*}_w \subseteq B^*_w$, so that $A^{1*}_w = B^*_w$. We then have the following situation:
(a): $B_w \subseteq A^1_w$ implies that the inclusion map $G : B_w \to A^1_w$ is an open map.
(b): $\|f\|_{A^1_w} \le M \|f\|_{B_w}$ implies that G is a bounded linear map.

Thus by the Open Mapping Theorem, the range of $G(B_w) = B_w$ is dense in A^1_w.
(c) Since $A^{1*}_w = B^*_w$, it follows that B_w and A^1_w are norm-equivalent, see for example [9] page 160.

□

Proof of Theorem 2.2. In the proof that $\|\cdot\|_{B_w}$ is a norm, only the triangle inequality requires special care. Using the definition of the infimum, let $\epsilon > 0$ and let $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}$ such that $f(\xi) = \sum_{n\in\mathbb{N}} \alpha_n b_{w,n}(\xi)$ and $g(\xi) = \sum_{n\in\mathbb{N}} \beta_n b_{w,n}(\xi)$ and $\sum_{n\in\mathbb{N}} |\alpha_n| < \|f\|_{B_w} + \epsilon/2$, $\sum_{n\in\mathbb{N}} |\beta_n| < \|g\|_{B_w} + \epsilon/2$. Hence $(f+g)(\xi) = \sum_{n\in\mathbb{N}} (\alpha_n + \beta_n) b_{w,n}(\xi)$ with $\sum_{n\in\mathbb{N}} |\alpha_n + \beta_n| \le \sum_{n\in\mathbb{N}} |\alpha_n| + \sum_{n\in\mathbb{N}} |\beta_n| < \infty$. Therefore,

$$\|f+g\|_{B_w} \le \sum_{n\in\mathbb{N}} |\alpha_n + \beta_n| \le \sum_{n\in\mathbb{N}} |\alpha_n| + \sum_{n\in\mathbb{N}} |\beta_n| < \|f\|_{B_w} + \|g\|_{B_w} + \epsilon \,.$$

Since ϵ is arbitrary, it follows that $\|f+g\|_{B_w} \le \|f\|_{B_w} + \|g\|_{B_w}$.
Now, let us prove that B_w is a Banach space. It will be sufficient to show that every absolutely convergent sequence is convergent. In short, it will be enough to show that given a sequence $\{f_n\}_{n\in\mathbb{N}}$, we have $\left\| \sum_{n\in\mathbb{N}} f_n \right\|_{B_w} \le \sum_{n\in\mathbb{N}} \|f_n\|_{B_w}$.
Let $\epsilon > 0$. Given $n \in \mathbb{N}$, there is a sequence α_{n_k} of real numbers such that $f_n(\xi) =$

$\sum_{k\in\mathbb{N}} \alpha_{n_k} b_{w,n_k}(\xi)$ with $\sum_{k\in\mathbb{N}} |\alpha_{n_k}| < \|f_n\|_{B_w} + \frac{\epsilon}{2^n}$. Therefore

$$\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\alpha_{n_k}| < \sum_{n\in\mathbb{N}} \|f_n\|_{B_w} + \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^n} = \sum_{n\in\mathbb{N}} \|f_n\|_{B_w} + \epsilon \, .$$

Since ϵ is arbitrary, it follows that

$$\left\| \sum_{n\in\mathbb{N}} f_n \right\| \leq \sum_{n\in\mathbb{N}} \|f_n\|_{B_w} \, .$$

□

Proof of Lemma 3.3. Let $P'_j(z_j,\xi_j) = \dfrac{\partial P_j(z_j,\xi_j)}{\partial z_j} = \dfrac{e^{i\xi_j}}{(e^{i\xi_j}-z_j)^2}$, for $j = 1, 2, \ldots, d$.
Let us start with $d = 2$. Let $J = [a_1 - h_1, a_1 + h_1] \times [a_2 - h_2, a_2 + h_2]$.
Then

$$F(z_1, z_2) = \frac{1}{(2\pi)^2} \int_J P_1(z_1,\xi_1) P_2(z_2,\xi_2) b_w(\xi_1,\xi_2) d\xi_1 d\xi_2,$$

and

$$\begin{aligned} f_1(z_1,z_2) &= \frac{2}{(2\pi)^2} \int_J P'_1(z_1,\xi_1) P_2(z_2,\xi_2) b_w(\xi_1,\xi_2) d\xi_1 d\xi_2 \\ &= \frac{2}{w(J)(2\pi)^2} [I_1 + I_2 - I_2 - I_4] \, , \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{a_1-h_1}^{a_1} \int_{a_2}^{a_2+h_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2, \quad I_2 = \int_{a_1}^{a_1+h_1} \int_{a_2-h_2}^{a_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2 \\ I_3 &= \int_{a_1-h_1}^{a_1} \int_{a_2-h_2}^{a_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2, \quad I_4 = \int_{a_1}^{a_1+h_1} \int_{a_2}^{a_2+h_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2 \, . \end{aligned}$$

Therefore

$$\begin{aligned} I_1 - I_4 &= \int_{a_1-h_1}^{a_1} \int_{a_2}^{a_2+h_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2 - \int_{a_1}^{a_1+h_1} \int_{a_2}^{a_2+h_2} P'_1(z_1,\xi_1)P_2(z_2,\xi_2) d\xi_1 d\xi_2 \\ &= -K_1(a_1,h_1,z_1) M_2(a_2,h_2,z_2) \, , \end{aligned}$$

where

$$\begin{aligned} K_1(a_1,h_1,z_1) &= - \int_{a_1-h_1}^{a_1} \frac{e^{i\xi_1}}{(e^{i\xi_1}-z_1)^2} d\xi_1 + \int_{a_1}^{a_1+h_1} \frac{e^{i\xi_1}}{(e^{i\xi_1}-z_1)^2} d\xi_1 \\ &= \frac{1}{i} \left[\frac{1}{z_1 - e^{i(a_1-h_1)}} + \frac{1}{z_1 - e^{i(a_1+h_1)}} + \frac{2}{e^{ia_1} - z_1} \right] \, . \end{aligned}$$

Also,

$$M_2(a_2,h_2,z_2) = \int_{a_2}^{a_2+h_2} P_2(z_2,\xi_2) d\xi_2$$

$$= \frac{1}{i}\left[-ih_2 + 2\ln(e^{ia_2} - z_2) - 2\ln(e^{i(a_2-h_2)} - z_2)\right] \ .$$

Likewise, we have $I_1 - I_3 = K_1(a_1, h_1, z_1)M_2'(a_2, h_2, z_2)$ with

$$M_2'(a_2, h_2, z_2) = \frac{1}{i}\left[-ih_2 - 2\ln(e^{ia_2} - z_2) + 2\ln(e^{i(a_2-h_2)} - z_2)\right] \ .$$

It follows that

$$I_1 - I_4 + I_2 - I_3 = K_1(a_1, h_1, z_1)\left[M_2'(a_2, h_2, z_2) - M_2(a_2, h_2, z_2)\right] = K_1(a_1, h_1, z_1)K_2 a_2, h_2, z_2),$$

where

$$K_2(a_2, h_2, z_2) = \frac{2}{i}\left[\ln(e^{i(a_2-h_2)} - z_2) + \ln(e^{i(a_2+h_2)} - z_2) - 2\ln(e^{ia_2} - z_2)\right] \ .$$

Hence

$$f_1(z_1, z_2) = C(J)K_1(a_1, h_1, z_1)K_2(a_2, h_2, z_2),$$

where $C(J) = \dfrac{2}{w(J)(2\pi)^2}$. Similarly, we obtain

$$f_2(z_1, z_2) = C(J)K_1(a_2, h_2, z_2)K_2(a_1, h_1, z_1) \ .$$

For $d \geq 2$, we observe that the constant $C(J)$ will remain the same, regardless of the variable of differentiation. Moreover, the function K_1 takes as arguments a_j, h_j, z_j if we are differentiating with respect to z_j and K_2 takes as arguments a_l, h_l, h_l, for all $l \neq j$. The product comes from the fact that the integrand is made of functions with separable variables. Hence, we conclude that

$$f_j(\boldsymbol{z}) = C(J)K_1(a_j, h_j, z_j)\prod_{\substack{l=1 \\ l\neq k}}^{d} K_2(a_l, h_l, z_l) \ .$$

Moreover,

$$|K_2(a_l, h_l, z_l)| \leq 2\left[\left|\ln(e^{i(a_l-h_l)} - z_l)\right| + \left|\ln(e^{i(a_l+h_l)} - z_l)\right| + 2\left|\ln(e^{i(a_l)} - z_l)\right|\right] \ .$$

Put $Z_{1l} = e^{i(a_l-h_l)} - z_l$. We know that $|z_l| < 1$, thus $\ln(|Z_{il}|) \leq \ln(2)$.

$$|\ln(Z_{1l})| = \sqrt{(\ln(|Z_{il}|))^2 + \arg(Z_{1l})^2}.$$

Consequently, $|\ln(Z_{1l})| \leq \sqrt{(\ln(2))^2 + \frac{\pi^2}{4}}$. Applying a similar argument to $\left|\ln(e^{i(a_l-h_l)} - z_l)\right|$ and $\left|\ln(e^{i(a_l-h_l)} - z_l)\right|$, we obtain that

$$|K_2(a_l, h_l, z_l)| \leq 3\sqrt{4(\ln(2))^2 + \pi^2} \ .$$

We will use a similar argument to [3] to deal with $K_1(a_1, h_1, z_1)$. However, this approach is much general than theirs in that they assumed that $a_1 = 0$ which is not assumed here. We observe that

$$iK_1(a_1, h_1, z_1) = \frac{1}{z_1 - e^{i(a_1-h_1)}} + \frac{1}{z_1 - e^{i(a_1+h_1)}} + \frac{1}{e^{ia_1} - z_1}$$

$$= \frac{2e^{ia_1}(z_1+e^{ia_1})(1-\cos h_1)}{(z_1-e^{i(a_1-h_1)})(z_1-e^{i(a_1+h_1)})(e^{ia_1}-z_1)} .$$

We have that

$$(z_1-e^{i(a_1-h_1)})(z_1-e^{i(a_1+h_1)}) = ((e^{ia_1}-z_1)^2+2e^{ia_1}z_1(1-\cos h_1),$$

so the modulus of the denominator of $iK_1(a_1,h_1,z_1)$ is

$$\begin{aligned}\left|(z_1-e^{i(a_1-h_1)})(z_1-e^{i(a_1+h_1)})(e^{ia_1}-z_1)\right| &= \left|(e^{ia_1}-z_1)^2+2e^{ia_1}z_1(1-\cos h_1)\right|\left|e^{ia_1}-z_1\right| \\ &\geq \left|\left(e^{ia_1}-z_1\right)^2-h_1^2\right|\left|e^{ia_1}-z_1\right| .\end{aligned}$$

The last inequality is obtained by noticing that $|z_1|<1, |e^{ia_1}|=1$, and $1-\cos h_1 \leq \frac{h_1^2}{2}$. Now consider $D_1=\{z_1\in\mathbb{D} : |e^{ia_1}-z_1|>2h_1\}$. For $z_1\in D_1$, the last inequality implies that

$$\left|(z_1-e^{i(a_1-h_1)})(z_1-e^{i(a_1+h_1)})(e^{ia_1}-z_1)\right| \geq \frac{3}{4}\left|e^{ia_1}-z_1\right|^3 \geq 6h_1^3.$$

On the other hand, the modulus of the numerator of $iK_1(a_1,h_1,z_1)$ is bounded by $2h_1^2$ on D_1, so that on D_1, one has

$$|K_1(a_1,h_1,z_1)| \leq \frac{8}{3}\frac{h^2}{|e^{ia_1}-z_1|^3} \leq \frac{1}{3h_1} . \tag{3.4}$$

Now let $z_1\in D_1^c$. Then we have that

$$\left|z_1-e^{i(a_1-h_1)}\right|, \left|z_1-e^{i(a_1+h_1)}\right|, \left|e^{ia_1}-z_1\right| \leq 4h_1 .$$

Put

$$\Phi_n^* = \left\{z_1\in\mathbb{D} : 2^{1-n}h_1 < \left|e^{ia_1}-z_1\right| \leq 2^{2-n}h_1\right\} = \left\{z_1\in\mathbb{D} : \frac{2^{n-2}}{h_1} \leq \frac{1}{|e^{ia_1}-z_1|} < \frac{2^{n-1}}{h_1}\right\} .$$

We note that $(0,1]=\bigcup\limits_{n=0}^{\infty}\left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right]$. It follows that

$$\begin{aligned} D_1^c &\subseteq \left\{z_1\in\mathbb{D} : \left|e^{ia_1}-z_1\right| \leq 4h_1\right\} \\ &\subseteq \bigcup_{n=0}^{\infty}\left\{z_1\in\mathbb{D} : \frac{4h_1}{2^{n+1}} < \left|e^{ia_1}-z_1\right| \leq \frac{4h_1}{2^n}\right\} \\ &= \bigcup_{n=0}^{\infty}\Phi_n^* .\end{aligned} \tag{3.5}$$

Thus, there exists an integer n such that

$$|K_1(a_1,h_1,z_1)| \leq \frac{1}{|z_1-e^{i(a_1-h_1)}|}+\frac{1}{|z_1-e^{i(a_1+h_1)}|}+\frac{1}{|e^{ia_1}-z_1|} \leq 3\frac{2^{n-1}}{h_1} .$$

We conclude by taking $C_1=\max\{3\frac{2^{n-1}}{h_1},\frac{1}{3h_1}\}$ and $C_2=3\sqrt{4(\ln(2))^2+\pi^2}$. □

Proof of Lemma 3.4.

Part (a): Put $J = [a-h, a+h]$ for some real numbers a and $h > 0$. Let N be the smallest integer such that $2^N h \geq 1$. Then for all $n \leq N$, and $z \in D_1$, we have $2^n h < |e^{ia} - z| < 2 < 2^{n+1}h$.
Also we observe that $\{z = re^{i\xi} \in \mathbb{D} : |e^{ia} - z| \leq \nu\} \subseteq \{z = re^{i\xi} \in \mathbb{D} : r \geq 1-\nu,\ |\xi - a| \leq \nu\}$.
Put $\Phi_n = \{z \in \mathbb{D} :\ 2^n h \leq |e^{ia} - z| \leq 2^{n+1}h\}$ and

$$U_1 = \int\int_{D_1} K_1(a,h,z) w(\xi) d\xi dr, \quad U_2 = \int\int_{D_1^c} K_1(a,h,z) w(\xi) d\xi dr \ .$$

It follows that

$$\begin{aligned}
U_1 &\leq \frac{8h^2}{3}\sum_{n=0}^{N}\int\int_{\Phi_n}\frac{w(\xi)}{|e^{ia}-z|^3}d\xi dr \\
&\leq \frac{8h^2}{3}\sum_{n=0}^{N}\frac{1}{(2^n h)^2}\int_{Q_{2^{n+1}h}(a)} w(\xi)d\xi \\
&\leq C\sum_{n=0}^{N}\frac{1}{(2^n h)^2}\int_{Q_{2^n h}(a)} w(\xi)d\xi, \quad \text{since } w \text{ is doubling} \\
&\leq C\sum_{n=0}^{N}\int_{2^n h \leq |\xi - a| \leq 2^{n+1}h}\frac{w(\xi)}{(\xi-a)^2}d\xi = C\int_h^{2^{N+1}h}\frac{w(\xi)}{(\xi-a)^2}d\xi \\
&\leq C\frac{w(J)}{|I|^2}\left(\frac{|J|^2}{w(J)}\int_{\xi\notin J}\frac{w(\xi)}{(\xi-a)^2}d\xi\right) < Cw(J), \quad \text{since } w \in \mathscr{B}_2
\end{aligned}$$

For $z \in D_1^c$, we have

$$\begin{aligned}
\int\int_{D_1^c}\frac{w(\xi)}{|e^{ia}-z|}d\xi dr &\leq \sum_{n=0}^{\infty}\int\int_{\Phi_n^*}\frac{w(\xi)}{|e^{ia}-z|}d\xi dr \quad \text{using equation (3.5)} \\
&\leq C\sum_{n=0}^{\infty}\frac{2^n}{h}\int_{1-2^{2-n}h}^{1}\int_{Q_{2^{2-n}h}(a)} w(\xi)d\xi dr \quad \text{using again equation (3.5)} \\
&\leq C\sum_{n=0}^{\infty}\int_{Q_{2^{1-n}h}(a)} w(\xi)d\xi, \quad \text{since } w \text{ is a doubling} \\
&\leq C\int_{Q_{2h}(a)} w(\xi)d\xi \leq C\int_{Q_h(a)} w(\xi)d\xi = Cw(J) \ .
\end{aligned}$$

It follows that for $z \in D_1^c$,

$$U_2 \ \leq \ Cw(J) \ .$$

Part (b): Suppose $w \in \mathcal{D}_1 \cap \mathcal{B}_2$ such that $w(\xi, r) \equiv \frac{w(1-r)}{1-r}$. Put

$$V_1 = \int\int_{D_1} K_1(a,h,z)\frac{w(1-r)}{1-r}drd\xi, \quad V_2 = \int\int_{D_1^c} K_1(a,h,z)\frac{w(1-r)}{1-r}drd\xi \ .$$

Using equation (3.4) above, we have that

$$\begin{aligned}
V_1 &\leq \frac{8h^2}{3}\int\int_{D_1}\frac{1}{|e^{ia}-z|^3}\frac{w(1-r)}{1-r}drd\xi\\
&\leq \frac{8h^2}{3}\sum_{n=0}^{N}\frac{1}{(2^nh)^3}\int_{-|a|-2^{n+1}h}^{|a|+2^{n+1}h}\int_{1-2^{n+1}h}^{1}\frac{w(1-r)}{1-r}drd\xi\\
&\leq \frac{16h^2}{3}\sum_{n=0}^{N}\left(\frac{|a|}{(2^nh)^3}+\frac{1}{(2^nh)^2}\right)\int_0^{2^{n+1}h}\frac{w(u)}{u}du, \quad \text{by change of variable } u=1-r\\
&\leq \frac{16h^2C}{3}\sum_{n=0}^{N}\left(\frac{|a|}{(2^nh)^3}+\frac{1}{(2^nh)^2}\right)w(2^nh), \quad \text{since } w\in\mathcal{D}_1\\
&= C(S_{11}+S_{12})
\end{aligned}$$

On one hand

$$\begin{aligned}
S_{11} &= \sum_{n=0}^{N}\frac{1}{(2^nh)^2}w(2^nh)\\
&= \sum_{n=0}^{N}\int_{a-2^{n+1}h}^{a+2^{n+1}h}\frac{w(2^nh)}{(2^nh)^3}du \leq C\sum_{n=0}^{N}\int_{a-2^{n+1}h}^{a+2^{n+1}h}\frac{w(u)}{(u)^3}du, \quad \text{since } w \text{ increasing and } u^3<8(2^nh)^3\\
&\leq C\int_{a-h}^{a+h}\frac{w(u)}{u^3}du \leq C\int_{a-h}^{1}\frac{w(u)}{u^3}du \leq \quad \text{since } J\subseteq[0,1]\\
&\leq C\frac{w(a-h)}{(a-h)^2} = Chw(a-h), \quad \text{since } w\in\mathcal{B}_2\\
&\leq Cw(J), \quad \text{since } w \text{ is increasing .}
\end{aligned}$$

On the other hand, we know that $\mathcal{D}_1\cap\mathcal{B}_2\subseteq\mathscr{D}=\bigcup_{p>1}\mathscr{B}_p$. Let $p>1$ such that $w\in\mathscr{B}_p$.

$$\begin{aligned}
S_{12} &= \sum_{n=0}^{N}\frac{|a|}{(2^nh)^3}w(2^nh)\\
&\leq |a|(N+1)\left(\sum_{n=0}^{N}\frac{1}{(2^nh)^3}\right)w(2^Nh), \quad \text{since } w \text{ is increasing}\\
&\leq C\frac{w(2^Nh)}{(2^Nh)^{p-1}} \quad \text{where } C=|a|(N+1)(2^Nh)^{p-1}\left(\sum_{n=0}^{N}\frac{1}{(2^nh)^3}\right)\\
&\leq C\int_{2^Nh}^{2^{N+1}h}\frac{w(2^Nh)}{(2^Nh)^p}du\\
&\leq 2^pC\int_{2^Nh}^{2^{N+1}h}\frac{w(u)}{u^p}du \quad \text{since } w \text{ is increasing}\\
&\leq 2^pC\frac{w(J)}{|J|^p}\left(\frac{|J|^p}{w(J)}\int_{u\notin J}\frac{w(u)}{u^p}du\right) < Cw(J), \quad \text{since } 2^Nh\geq 1 \text{ and } w\in\mathscr{B}_p.
\end{aligned}$$

Now

$$
\begin{aligned}
V_2 &\leq \sum_{n=0}^{\infty} \int\int_{\Phi_n^*} \frac{1}{|e^{ia}-z|}\frac{w(1-r)}{1-r} d\xi dr \\
&\leq \sum_{n=0}^{\infty} \int_{1-2^{2-n}h}^{1} \int_{-|a|-2^{2-n}h}^{|a|+2^{2-n}h} \frac{1}{|e^{ia}-z|}\frac{w(1-r)}{1-r} d\xi dr \\
&\leq \sum_{n=0}^{\infty} \frac{2^{n-1}}{h} \int_{1-2^{2-n}h}^{1} \int_{-|a|-2^{2-n}h}^{|a|+2^{2-n}h} \frac{w(1-r)}{1-r} d\xi dr, \quad \text{since } \tfrac{1}{|e^{ia}-z|} < \tfrac{2^{n-1}}{h} \text{ on } \Phi_n^* \\
&\leq \sum_{n=0}^{\infty} \frac{2^n}{h}(|a|+2^{2-n}h) \int_0^{2^{2-n}h} \frac{w(u)}{u} du, \quad \text{after the change of variable } u = 1-r \\
&= S_{21} + S_{22}\ .
\end{aligned}
$$

On one hand,

$$
\begin{aligned}
S_{21} &= \sum_{n=0}^{\infty} \frac{2^n}{h} 2^{2-n}h \int_0^{2^{2-n}h} \frac{w(u)}{u} du = 4\sum_{n=0}^{\infty} \int_0^{2^{2-n}h} \frac{w(u)}{u} du = 4\int_0^{4h} \frac{w(u)}{u} du \\
&\leq Cw(4h) \leq Cw(h) \leq Cw(J), \quad \text{since } w \in \mathcal{D}_1 \cap \mathcal{B}_2 \subseteq \mathscr{D}\ .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
S_{22} &= |a| \sum_{n=0}^{\infty} \frac{2^n}{h} \int_0^{2^{2-n}h} \frac{w(u)}{u} du \\
&\leq C\sum_{n=0}^{\infty} \frac{2^n}{h} w(4\cdot 2^{-n}h), \quad \text{since } w \in \mathcal{D}_2 \\
&\leq C\sum_{n=0}^{\infty} \frac{2^n}{h} w(2^{-n}h), \quad \text{since } w \in \mathscr{D} \\
&\leq C\sum_{n=0}^{\infty} \int_{2^{-n}h}^{2^{1-n}h} \frac{w(u)}{u} du, \quad \text{since } \tfrac{1}{2^{1-n}h} \leq \tfrac{1}{u} \leq \tfrac{1}{2^{-n}h} \\
&\leq C\int_0^{2h} \frac{w(u)}{u} du \leq Cw(2h) \leq Cw(J), \quad \text{since } w \in \mathscr{D}.
\end{aligned}
$$

□

Proof of Lemma 3.5.
We will start again with the case $d = 2$. We know from above that

$$
iK_1(a_1, h_1, z_1) = \frac{2e^{ia_1}(z_1 + e^{ia_1})(1-\cos h_1)}{(e^{ia_1} - z_1)((e^{ia_1} - z_1)^2 + 2e^{ia_1} z_1 (1-\cos h_1))}\ .
$$

Let us choose h_1 and a_1 so that $h_1 < |e^{ia_1} - z_1|$. Then $|z_1 + e^{ia_1}| \geq 2 - h_1$. Since $1 - \frac{h_1^2}{2} \leq \cos h_1 \leq 1 - \frac{h_1^2}{2} + \frac{h_1^4}{24}$, we obtain

$$\begin{aligned} |K_1(a_1, h_1, z_1)| &\geq \frac{(2-h_1)\left(h_1^2 - \frac{h_1^4}{12}\right)}{|(e^{ia_1} - z_1)|\,|(e^{ia_1} - z_1)^2 + 2e^{ia_1} z_1(1-\cos h_1)|} \\ &\geq \frac{(2-h_1)\left(h_1^2 - \frac{h_1^4}{12}\right)}{|e^{ia_1} - z_1|^3} = \frac{Ch_1^2}{|e^{ia_1} - z_1|^3} \, . \end{aligned} \tag{3.6}$$

Now let us choose a_2 and h_2 such that $1 < h_2 < |e^{ia_2} - z_2|$. We also know from above that

$$K_2(a_2, h_2, z_2) = \frac{2}{i}\left[\ln(e^{i(a_2-h_2)} - z_2) + \ln(e^{i(a_2+h_2)} - z_2) - 2\ln(e^{ia_2} - z_2)\right] \, .$$

We know that $|\ln(e^{ia_2} - z_2)| \geq \ln(|(e^{ia_2} - z_2)|) \geq \ln(h_2) > 0$. Therefore

$$|K_2(a_2, h_2, z_2)| \geq 4\ln(h_2) \, . \tag{3.7}$$

Combining both equations (3.6) and (3.7), it follows that for $h_1 < |e^{ia_1} - z_1|$ and $1 < h_2 < |e^{ia_2} - z_2|$

$$|f_1(z_1, z_2)| \geq \frac{C(h_1)h_1^2}{|e^{ia_1} - z_1|^3} \, .$$

Now for $d \geq 2$, we can generalize it so that given $1 \leq j \leq d$ and $l \neq j$ one has

$$|f_j(\boldsymbol{z})| \geq \frac{C(h_j)h_j^2}{|e^{ia_j} - z_j|^3}, \quad \text{for} \quad h_j < \left|e^{ia_j} - z_j\right|, \quad 1 < h_l < \left|e^{ia_l} - z_l\right| \, . \tag{3.8}$$

Assume that $z_j = r_i e^{i\xi_j}$ with $r_j < 1$. Then

$$\begin{aligned} \left|e^{ia_j} - z_j\right|^2 &= 1 - r_j^2 - 2r_j\cos(\xi_j - a_j) \\ &\leq (1-r_j)^2 + (\xi_j - a_j)^2 \, . \end{aligned}$$

Let $D_j = \{z_j \in \mathbb{D} : h_j < |e^{ia_j} - z_j|\}$ and $D_j^* = \{z_j \in \mathbb{D} : h_j < \xi_j < a_j + 1\}$. Note that then $D_j \cap D_j^* \neq \emptyset$. We have that

$$\begin{aligned} \int\int_{\mathbb{D}} |f_j(\boldsymbol{z})|\, w_j(\xi) d\xi_j dr_j &\geq h_j^2 \int\int_{D_j} \frac{w_j(\xi_i)}{|e^{ia_j} - z_j|^3} d\xi_j dr_j \\ &\geq h_j^2 \int_0^1 \int_{\xi > h_j} \frac{w_j(\xi_i)}{((1+r_j)^2 + (\xi_j - a_j)^2)^{3/2}} d\xi_j dr_j \\ &\geq h_j^2 \int_{\xi > h_j} \frac{w_j(\xi_i)}{(\xi_j - a_j)^2} d\xi_j \quad \text{since } x^{-3} > x^{-2} \text{ on } (0,1) \\ &\geq h_j^2 \int_{\xi > h_j} \frac{w_j(\xi_i)}{\xi_j^2} d\xi_j \quad \text{since } 0 < h_j < \xi_j < a_j + 1 \, . \end{aligned}$$

□

Proof of Lemma 3.6. Firstly, let

$$D_{1j} = \left\{ z_j \in \mathbb{D} : h_j < \left| e^{ia_j} - z_j \right| \right\}.$$

Therefore, if $1 - r_j < |\xi_j - a_j|$, we have $h_j \leq |e^{ia_j} - z_j| \leq \sqrt{2}(\xi_j - a_j)$. There are two possibilities: either $1 - r_j < h_j$ or $1 - r_j \geq h_j$. So let us consider the subset

$$D^*_{1j} = \{ z_j \in \mathbb{D} : h_j \leq 1 - r_j < |\xi_j - a_j| \}$$

of D_{1j}. It follows from equation (3.8) that

$$\begin{aligned}
\int\int_{\mathbb{D}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j &\geq \int\int_{D_{2j}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \\
&\geq C(h_j) \int\int_{D_{1j}} \frac{h_j^2}{|e^{ia_j} - z_j|^3} \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \\
&\geq \frac{h_j^2 C(h_j)}{4\sqrt{2}} \int\int_{D_{1j}} \frac{1}{(\xi_j - a_j)^3} \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \\
&= \frac{h_j^2 C(h_j)}{4\sqrt{2}} \int_0^{1-h_j} \frac{w_j(1-r_j)}{1-r_j} \left(\int_{1-r_j+a_j}^{\pi} \frac{1}{(\xi_j - a_j)^3} d\xi_j \right) dr_j \\
&\geq C_j \int_{h_j}^1 \frac{w_j(u_j)}{u_j^3} du_j .
\end{aligned}$$

The latter inequality is obtained after the changes of variable $\xi - a = \theta$ and $1 - r = u$ respectively, with $C_j = \dfrac{h_j^2 C(h_j)}{\sqrt{2}}$.

Secondly, consider

$$D_{2j} = \left\{ z_j \in \mathbb{D} : \left| e^{ia_j} - z_j \right| \leq \frac{h_j}{4} \right\}.$$

As above, if $|\xi_j - a_j| < 1 - r_j$, we have $|e^{ia_j} - z_j| \leq \sqrt{2}(1 - r_j)$. So either $\sqrt{2}(1-r_j) < \frac{h_j}{4}$ or $\sqrt{2}(1 - r_j) \geq \frac{h_j}{4}$. Thus, consider the subset

$$D^*_{2j} = \left\{ z_j \in \mathbb{D} : |\xi_j - a_j| < 1 - r_j < \frac{h_j}{4\sqrt{2}} \right\}$$

of D_{2j}. Therefore, we have

$$\begin{aligned}
\int\int_{\mathbb{D}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j &\geq \int\int_{D_{2j}} |f_j(\boldsymbol{z})| \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \\
&\geq C(h_j) \int\int_{D_{2j}} \frac{h_j^2}{|e^{ia_j} - z_j|^3} \frac{w_j(1-r_j)}{1-r_j} d\xi_j dr_j \\
&= \frac{C(h_j)}{16\sqrt{2}} \int_0^{1-\frac{h_j}{4\sqrt{2}}} \frac{w_j(1-r_j)}{(1-r_j)^2} \left(\int_{a_j+r_j-1}^{a_j+1-r_j} d\xi_j \right) dr_j \\
&= C_j \int_0^{\frac{h_j}{4\sqrt{2}}} \frac{w_j(u_j)}{u_j} du_j,
\end{aligned}$$

after the change of variable $u = 1 - r$ with $C_j = \dfrac{C(h_j)}{16\sqrt{2}}$. □

4 Conclusion

We have proposed in this paper a space that acts as the analytic extension of the so-called special atom spaces in higher dimensions. What we find interesting and remarkable in these spaces is their apparent simplicity. Certainly one could think of atoms defined on intervals that are different from the characteristic functions on these intervals, say for instance polynomial type of atoms. However, the properties like orthonormality would be difficult to prove, especially in higher dimensions. The results in this paper open the door to exploring lacunary sequences in Bergman-Besov-Lipschitz spaces in higher dimensions. Another idea that is no more far-fetched in the idea of Blaschke-products in these spaces, that are non-complex in their inception but have complex extensions.

5 Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] N. Arcozzi, R. Rochberg, and E. Sawyer. Carleson measures and interpolating sequences for besov spaces on complex balls. *Mem. Amer. Math. Soc.*, 186 (559), 2006. vi+163.

[2] A. Beurling. On two problems concerning linear transformations in hilbert space. *Acta Math.*, 81 (17), 1948.

[3] S. Bloom and G. De Souza. Atomic decomposition of generalized lipschitz spaces. *Illinois Journal of Mathematics*, pages 682–686, 1989.

[4] A. Brandigneres, E. Fricain, F. Gaunard, A. Hartman, and W. T. Ross. Direct and reverse carleson measures for $\mathscr{H}(b)$ spaces. *Indiana Univ. Math. J.*, 64 (4):1027–1057, 2015.

[5] R. Coifman. A real variable characterization of h^p. *Studia Mathematica*, 51 :269–274, 1974.

[6] G. De Souza. The atomic decomposition of bergman-besov-lipschitz spaces. *Proc. Amer. Math. Soc.*, 14 :682–686, 1985.

[7] G. De Souza and G. Sampson. function in the dirichlet space such that its fourier series divergesalmost everywhere. *Proceedings of the American Mathematical Society*, 120 (3):723–726, 1994.

[8] C. Fefferman. Characterization of bounded mean oscillation. *Bulletin of the American Mathematical Society*, 77 (4):587–588, 1971.

[9] G. G. Folland. *Real analysis, modern techniques abd their applications.* Wiley-Interscience, 1999.

[10] E. Fricain and J. Mashreghi. *New Mathematical Monographs*, volume 1 of *xix+681 pp.* Cambridge University Press, 2016.

[11] B.-E. Idrissi and O. El Farah. Blaschke products and zero sets in weighted dirichlet spaces. *Potential Analysis*, 2020.

[12] E. Kwessi, G. De Souza, A. Abebe, and R. Aulaskari. Characterization of lacunary functions in bergman-besov-lipschitz spaces. *Complex Variables and Elliptics Equations*, 58 (2):157–162, 2013.

[13] E. Kwessi, G. De Souza, D. Ngalla, and N. Mariama. The special atom space in higher dimensions. *Demonstratio Mathematica*, 53 :131–151, 2020.

[14] B. Muckenhoupt. Weighted inequalities for the hardy maxima operator. *Transactions of the American Mathematical Society*, 165 :107–226, 1972.

[15] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 1987.

[16] K. Zhu. Bloch spaces of analytic functions. *Rocky Mountain Journal of Mathematics*, 23 (3):1143–1177, 1993.