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Research Article

A Note on Multiplication and Composition Operators in Lorentz Spaces

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we revisit the Lorentz spaces $L(p, \mathrm{q})$ for $p > 1, q > 0$ defined by G. G. Lorentz in the nineteen fifties and we show how the atomic decomposition of the spaces $L(p, 1)$ obtained by De Souza in 2010 can be used to characterize the multiplication and composition operators on these spaces. These characterizations, though obtained from a completely different perspective, confirm the various results obtained by S. C. Arora, G. Datt and S. Verma in different variants of the Lorentz Spaces.

1. Introduction

In the early 1950s, Lorentz introduced the now famous Lorentz spaces $L(p, q)$ in his papers [1, 2] as a generalization of the L^p spaces. The parameters p and q encode the information about the size of a function; that is, how tall and how spread out a function is. The Lorentz spaces are quasi-Banach spaces in general, but the Lorentz quasi-norm of a function has better control over the size of the function than the L^p norm, via the parameters p and q, making the spaces very useful. We are mostly concerned with studying the multiplication and composition operators on Lorentz spaces. These have been studied before by various authors in particular by Arora et al. in [3–6]. In this paper, the results we obtain are in accordance with what these authors have found before. We believe that the techniques and relative simplicity of our approach are worth reporting to further enrich the topic. Our results, found on the boundary of the unit disc due to the original focus by De Souza in [7], will show how one can use the atomic characterization of the Lorentz space $L(p,1)$ in the study of multiplication and composition operators in the spaces $L(p,q)$.

2. Preliminaries

Let (X,μ) be a measure space.

Definition 2.1. Let f be a complex-valued function defined on X. *The decreasing rearrangement* of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\{y > 0 : d(f,y) \leq t\}, \tag{2.1}$$

where $d(f,y) = \mu(\{x : |f(x)| > y\})$ is the *distribution* of the function f.

Definition 2.2. Given a measurable function f on (X,μ) and $0 < p,\ q \leq \infty$, define

$$\|f\|_{L(p,q)} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \dfrac{dt}{t}\right)^{1/q}, & \text{if } q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & \text{if } q = \infty. \end{cases} \tag{2.2}$$

The set of all functions f with $\|f\|_{L(p,q)} < \infty$ is called *the Lorentz space* with indices p and q and denoted by $L(p,q)(X,\mu)$.

We now consider the measure μ on X to be finite. Let $g : X \to X$ be a μ-measurable function such that $\mu(g^{-1}(A)) \leq C\mu(A)$ for a μ-measurable set $A \subseteq [0,2\pi]$ and for an absolute constant C. Here $g^{-1}(A)$ refers to the preimage of the set A.

Remark 2.3. It is important to note that $\|g\| = \sup_{\mu(A)\neq 0}(\mu(g^{-1}(A))/\mu(A)\ )$ is not necessarily a norm.

Definition 2.4. For a given function g, we define the multiplication operator T_g on Lorentz spaces as $T_g(f) = f \cdot g$ and the composition operator C_g as $C_g(f) = f \circ g$.

The following two results are used in our proofs. The first is a result of De Souza [7] which gives an atomic decomposition of $L(p,1)$. The second is the Marcinkiewicz interpolation theorem (see [8]) which we state for completeness of presentation.

Theorem 2.5 (see De Souza [7]). *A function $f \in L(p,1)$ for $p > 1$ if and only if $f(t) = \sum_{n=1}^{\infty} c_n \chi_{A_n}(t)$ with $\sum_{n=1}^{\infty} |c_n| \mu^{1/p}(A_n) < \infty$, whereas μ is measure on X and A_n are μ-measurable sets in X. Moreover, $\|f\|_{L(p,1)} \cong \inf \sum_{n=1}^{\infty} |c_n| \mu^{1/p}(A_n)$, where the infimum is taken over all possible representations of f.*

Theorem 2.6 (see Marcinkiewicz). *Assume that for $0 < p_0 \neq p_1 \leq \infty$, for all $q > 0$, for all measurable subsets A of X, there are some constants $0 < M_0,\ M_1 < \infty$ such that for a linear or quasi-linear operator T_g*

(a) $\|T_g\chi_A\|_{L(p_0,\infty)} \leq M_0\mu^{1/p_0}(A)$.

(b) $\|T_g\chi_A\|_{L(p_1,\infty)} \leq M_1\mu^{1/p_1}(A)$.

Then there is some $M > 0$ such that $\|T_g f\|_{L(p,q)} \leq M\|f\|_{L(p,q)}$ for $1/p = \theta/p_0 + (1-\theta)/p_1$, $0 < \theta < 1$.

One implication of Theorem 2.5 is that it can be used to prove and justify a theorem of Stein and Weiss [9]. That is, to show that linear operators $T : L(p,1) \to B$ are bounded, where B is Banach space closed under absolute value and satisfying $\|f\|_B = \||f|\|_B$, all one needs to show is that $\|T\chi_A\|_B \leq M\mu^{1/p}(A),\ p > 1$. Theorem 2.6 will be used to show that valid results on $L(p,1)$ are also valid on $L(p,q)$.

Definition 2.7. We denote by M_r^p the set of real-valued functions defined on $X = [0,2\pi]$ such that

$$\|f\|_{M_r^p} = \sup_{x>0}\left(\frac{r}{px^{1/p}}\int_0^x \left(f^*(t)t^{1/p}\right)^r \frac{dt}{t}\right)^{1/r} < \infty, \tag{2.3}$$

where $1 \leq p \leq r < \infty$.

We will show that the space M_r^p is equivalent to a weak L^k space for some k that depends on p and r and $\|\cdot\|_{M_r^p}$ is quasinorm.

Lemma 2.8. *$\|\cdot\|_{M_r^p}$ a quasinorm on M_r^p.*

Proof. $f^* \geq 0$ by definition. This implies that $\|f\|_{M_r^p} \geq 0$. Moreover, $\|f\|_{M_r^p} = 0$ implies that for all $0 < x \leq 2\pi$, $\int_0^x (f^*(t)t^{1/p})^r dt/t = 0$. Hence, we have $f^* = 0$ μ-a.e, thus, $f = 0$ since f is a representative of an equivalence class. Now let $k \neq 0$ be a real constant, $f \in M_r^p$, and $x \in (0,2\pi]$. Noting $(kf)^* = |k|f^*$, the homogeneity condition $\|kf\|_{M_r^p} = |k|\|f\|_{M_r^p}$ follows trivially. Let $f, g \in M_r^p$. Since $(f+g)^*(t) \leq f^*(t/2) + g^*(t/2)$, for any $x \in (0,2\pi]$, we have

$$\begin{aligned}\int_0^x \left((f+g)^*(t)t^{1/p}\right)^r \frac{dt}{t} &\leq 2^{r-1}\left(\int_0^x \left(f^*\left(\frac{t}{2}\right)t^{1/p}\right)^r \frac{dt}{t} + \int_0^x \left(g^*\left(\frac{t}{2}\right)t^{1/p}\right)^r \frac{dt}{t}\right) \\ &\leq 2^{\frac{r}{p}+r-1}\left(\int_0^{(1/2)x} \left(f^*(t)t^{1/p}\right)^r \frac{dt}{t} + \int_0^{(1/2)x} \left(g^*(t)t^{1/p}\right)^r \frac{dt}{t}\right).\end{aligned} \tag{2.4}$$

Since $(a+b)^{1/r} \leq a^{1/r} + b^{1/r}$ for $a, b > 0$, we have

$$\|f+g\|_{M_r^p} \leq 2^{1/p-1/r+1}\left(\|f\|_{M_r^p} + \|g\|_{M_r^p}\right), \text{ with } 2^{1/p-1/r+1} > 1 \text{ for } r, p > 1. \quad (2.5)$$

□

Theorem 2.9. *$M_r^p \cong L(pr', \infty)$, where $r, r' \geq 1, 1/r + 1/r' = 1$.*

Proof. Suppose $g \in M_r^p$. There is an absolute constant C such that for all $x > 0$,

$$C \geq \left(\frac{r}{px^{1/p}}\int_0^x \left(g^*(t)t^{1/p}\right)^r \frac{dt}{t}\right)^{1/r} \geq \left(\frac{r}{px^{1/p}}\left(g^*(x)\right)^r \int_0^x t^{r/p-1}dt\right)^{1/r} = x^{1/pr'}g^*(x). \quad (2.6)$$

Thus, $\sup_{x>0} x^{1/pr'}g^*(x) \leq C$ implying that $g \in L(pr', \infty)$.

Conversely, let $g \in L(pr', \infty)$. Then there is an absolute constant C such that, $g^*(t) \leq Ct^{-1/pr'}$. This implies that $(g^*(t)t^{1/p})^r \leq C^r t^{1/p}$. Thus,

$$\sup_{x>0}\left(\frac{r}{px^{1/p}}\int_0^x \left(g^*(t)t^{1/p}\right)^r \frac{dt}{t}\right)^{1/r} \leq \sup_{x>0}\left(C^r \frac{r}{px^{1/p}}\int_0^x t^{1/p}\frac{dt}{t}\right)^{1/r} = Cr^{1/r}. \quad (2.7)$$

This implies that $g \in M_r^p$. □

Remark 2.10. One can easily see from Theorem 2.9 that $M_\infty^p \cong L(p, \infty)$ and $M_1^p \cong L^\infty$. Moreover, $\|g\|_{M_1^p} = \|g\|_\infty$. To see this, note that

$$\begin{gathered}\|g\|_{M_1^p} = \sup_{x>0}\left(\frac{1}{px^{1/p}}\int_0^x g^*(t)t^{1/p}\frac{dt}{t}\right) \leq \|g\|_\infty \sup_{x>0}\left(\frac{1}{px^{1/p}}\int_0^x t^{1/p-1}dt\right) = \|g\|_\infty, \\ \|g\|_{M_1^p} \geq \frac{g^*(x)}{px^{1/p}}\int_0^x t^{1/p-1}dt = g^*(x)\end{gathered} \quad (2.8)$$

for all x since g^* is decreasing. Taking the limit as $x \to 0$, we see that $\|g\|_{M_1^p} \geq g^*(0) = \|g\|_\infty$.

3. Main Results

3.1. Multiplication Operators

Theorem 3.1 (see multiplication operator on $L(p,1)$). *The multiplication operator $T_g : L(p,1) \to L(p',1)$ for $p' \geq p > 1$ is bounded if and only $g \in L^\infty$. Moreover, $\|T_g\| = \|g\|_\infty$.*

Proof. It is convenient to use M_1^p which is equivalent to L^∞. Assume that $= \|T_g f\|_{L(p',1)} \leq C\|f\|_{L(p,1)}$. Then for $f = \chi_{[0,x]}$ where $x \in (0, 2\pi]$,

$$\int_0^{2\pi} \left(T_g \chi_{[0,x]}\right)^*(t)t^{1/p'-1}dt = \int_0^x g^*(t)t^{1/p'-1}dt \leq C\int_0^{2\pi} \chi_{[0,x]}^*(t)t^{1/p-1}dt = Cpx^{1/p}. \quad (3.1)$$

Multiplying and dividing the integrand on the left by $t^{1/p-1}$, we get

$$\int_0^x g^*(t)t^{1/p-1}t^{p-p'/pp'}dt \leq Cpx^{1/p}. \tag{3.2}$$

Since $t \mapsto t^{(p-p')/pp'}$ is decreasing on $[0, x]$ and $0 < x \leq 2\pi$, we have

$$\frac{1}{px^{1/p}} \int_0^x g^*(t)t^{1/p-1}dt \leq C(2\pi)^{p-p'/pp'}. \tag{3.3}$$

Taking the supremum over all $x > 0$, we have that $g \in M_1^p$.

Assume that $g \in M_1^p$ and $x > 0$. Since $p' > p$ we have

$$\|T_g\chi_{[0,x]}\|_{L(p',1)} = \int_0^x g^*(t)t^{1/p'-1}dt \leq \int_0^x g^*(t)t^{1/p-1}dt. \tag{3.4}$$

And so,

$$\|T_g\chi_{[0,x]}\|_{L(p',1)} \leq M\|\chi_{[0,x]}\|_{L(p,1)}, \text{ where } M = \sup_{x>0}\frac{1}{px^{1/p}}\int_0^x g^*(t)t^{1/p-1}dt. \tag{3.5}$$

Using the atomic decomposition of $L(p, 1)$, we get

$$\|T_g f\|_{L(p',1)} \leq M'\|f\|_{L(p,1)} \quad \text{for some positive constant } M'. \tag{3.6}$$

To prove the second part of the theorem, first note that the expression in (3.5) gives that $\|T_g\| \leq \|g\|_\infty$. Now take $f = (1/x^{1/p})\ \chi_{[0,x]}$. We can easily see that $\|f\|_{L(p,1)} = 1$ and $\|T_g f\|_{L(p',1)} \geq g^*(x)$ for $x \in [0, 2\pi]$ since g^* is decreasing. Now taking the sup over $\|f\|_{L(p,1)} \leq 1$ and the limit as $x \to 0$ gives $\|T_g\| \geq \|g\|_\infty$. Thus, $\|T_g\| = \|g\|_\infty$. □

The following theorem, which is equivalent to Theorem 1.1 of [6], follows from Theorems 2.6 and 3.1.

Theorem 3.2 (see multiplication operator on $L(p, q)$). *The multiplication operator $T_g : L(p, q) \to L(p, q)$ is bounded if and only if $g \in L^\infty$ for $1 < p \leq \infty$, $1 < q \leq \infty$. Moreover, $\|T_g\| = \|g\|_\infty$.*

Remark 3.3. Since, by Theorem 2.9, $M_1^p \subsetneq M_r^p$ for $r > 1$, the theorem implies that if the multiplication operator $T_g : L(p, q) \to L(p, q)$ defined by $T_g f = g \cdot f$ is bounded, then $g \in M_r^p$ for $p, q > 1$.

Remark 3.4. It is worth observing that the norm

$$\|f\|_{L(p,1)} = \sup_{\substack{A\subseteq X \\ \mu(A)\neq 0}} \frac{1}{\mu^{1/p}(A)} \int_0^{\mu(A)} g^*(t)t^{1/p-1}dt \tag{3.7}$$

can also be used to prove the previous theorem, first on $L(p, 1)$ and then on $L(p, q)$ by means of either the Marcienkiewiecz Inteporlation Theorem or Theorem 2.5. Actually, this norm was the original motivation for the introduction of the space M_r^p. For sake of simplicity and without loss of generality, we modified it by replacing $\mu(A)$, $A \subset X$ by $x > 0$.

Theorem 3.5. *If $f \in L(p_1, q_1)$, and $g \in L(p_2, q_2)$, where $1 < p_1, p_2, q_1, q_2 < \infty$, then $g \cdot f \in L(r, s)$ where $1/r = 1/p_1 + 1/p_2$ and $1/s = 1/q_1 + 1/q_2$.*

Proof. Given $1 < p_1, p_2, q_1, q_2 < \infty$, assume $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$. Let r, s be such that $1/r = 1/p_1 + 1/p_2$ and $1/s = 1/q_1 + 1/q_2$. Since $(f \cdot g)^*(t) \le f^*(t)g^*(t)$, we have

$$\int_0^{2\pi} \left((f \cdot g)^*(t)t^{1/r}\right)^s \frac{dt}{t} \le \int_0^{2\pi} \left(f^*(t)t^{1/p_1}\right)^s \cdot \left(g^*(t)t^{1/p_2}\right)^s \frac{dt}{t}. \tag{3.8}$$

Using Holder's inequality on the RHS with $s/q_1 + s/q_2 = 1$, we have

$$\int_0^{2\pi} \left((f \cdot g)^*(t)t^{1/r}\right)^s \frac{dt}{t} \le \left(\int_0^{2\pi} \left(f^*(t)t^{1/p_1}\right)^{q_1} \frac{dt}{t}\right)^{s/q_1} \cdot \left(\int_0^{2\pi} \left(g^*(t)t^{1/q_2}\right)^{q_2} \frac{dt}{t}\right)^{s/q_2}. \tag{3.9}$$

Thus, we have

$$\|g \cdot f\|_{L(r,s)} \le \|f\|_{L(p_1,q_1)} \cdot \|g\|_{L(p_2,q_2)}. \tag{3.10}$$

□

Theorem 3.6. *If $g \in M_r^p$, then $T_g : L(q, s) \to L(pqr'/(pr'+q), s)$ is bounded, where $1/r + 1/r' = 1$ and for $s > 0$ and $p, q > 1$.*

Proof. Let $g \in M_r^p \cong L(pr', \infty)$

$$\begin{aligned} \|T_g f\|_{L(k,s)}^s &\le \int_0^{2\pi} \left(g^*(t) \cdot f^*(t)t^{1/k}\right)^s \frac{dt}{t} \\ &= \int_0^{2\pi} \left(g^*(t)t^{1/pr'} \cdot f^*(t)t^{1/q}\right)^s \frac{dt}{t} \quad \text{if } \frac{1}{k} = \frac{1}{q} + \frac{1}{qr'}. \end{aligned} \tag{3.11}$$

Therefore,

$$\|T_g f\|_{L(k,s)}^s \le \sup_{t>0}\left(g^*(t)t^{1/pr'}\right)^s \cdot \int_0^{2\pi} \left(f^*(t)t^{1/q}\right)^s \frac{dt}{t}. \tag{3.12}$$

That is,

$$\|T_g f\|_{L(k,s)} \le \|g\|_{M_r^p} \cdot \|f\|_{L(q,s)}, \quad \text{where } k = \frac{pr'q}{pr' + q}. \tag{3.13}$$

□

Noting that $M_r^p \cong L(pr', \infty)$, $r' = r/(r-1)$, it is easy to see that Theorem 3.6 shows that the result of Theorem 3.5 extends to the case where $q_2 = \infty$.

3.2. Composition Operators

Theorem 3.7. *The composition operator $C_g : L(p,q) \to L(p,q)$ is bounded if and only if there is an absolute constant C such that*

$$\mu\left(g^{-1}(A)\right) \leq C\mu(A), \tag{3.14}$$

for all μ-measurable sets $A \subseteq [0,2\pi]$ and for $1 < p \leq \infty, 1 \leq q \leq \infty$. Moreover, $\|C_g\| = \|g\|^{1/p}$.

Proof. We will prove this theorem for $L(p,1)$ and use the interpolation theorem to conclude for $L(p,q)$.

First assume that $C_g : L(p,1) \to L(p,1)$ is bounded that is, there is an absolute constant C such that

$$\|C_g f\|_{L(p,1)} \leq C\|f\|_{L(p,1)}. \tag{3.15}$$

Let A be a μ-measurable set in $[0,2\pi]$ and let $f = \chi_A$. Then, (3.15) is equivalent to

$$\begin{aligned} \|C_g\chi_A\|_{L(p,1)} \leq C\|\chi_A\|_{L(p,1)} \Longleftrightarrow \frac{1}{p}\int_0^{2\pi} (C_g\chi_A)^*(t)t^{1/p-1}dt \\ \leq \frac{C}{p}\int_0^{2\pi} \chi_A^*(t)t^{(1/p)-1}dt; \end{aligned} \tag{3.16}$$

that is,

$$\frac{1}{p}\int_0^{2\pi} (\chi_A \circ g)^*(t)t^{1/p-1}dt \leq \frac{C}{p}\int_0^{2\pi} \chi_{[0,\mu(A)]}(t)t^{(1/p)-1}dt. \tag{3.17}$$

Since $(\chi_A \circ g) = \chi_{g^{-1}(A)}$, then $(\chi_A \circ g)^* = \chi_{[0,\mu(g^{-1}(A))]}$. Therefore, the previous inequality gives

$$\frac{1}{p}\int_0^{\mu(g^{-1}(A))} t^{1/p-1}dt \leq \frac{C}{p}\int_0^{\mu(A)} t^{1/p-1}dt. \tag{3.18}$$

And hence,

$$\mu\left(g^{-1}(A)\right) \leq C^p\mu(A). \tag{3.19}$$

On the other hand, assume that there is some constant $C > 0$ such that $\mu(g^{-1}(A)) \leq C\mu(A)$. Then,

$$\begin{aligned}
\left\|C_g\chi_A\right\|_{L(p,1)} &= \frac{1}{p}\int_0^{2\pi} (\chi_A \circ g)^*(t)t^{1/p-1}dt \\
&= \frac{1}{p}\int_0^{2\pi} \chi_{[0,\mu(g^{-1}(A))]}(t)t^{1/p-1}dt \\
&= \left(\mu\left(g^{-1}(A)\right)\right)^{1/p} \leq C^{1/p}(\mu(A))^{1/p}.
\end{aligned} \tag{3.20}$$

Consequently,

$$\left\|C_g\chi_A\right\|_{L(p,1)} \leq C^{1/p}(\mu(A))^{1/p}. \tag{3.21}$$

As a consequence of Theorem 2.5 or the result by Weiss and Stein in [9], we have

$$\left\|C_g f\right\|_{L(p,1)} \leq C^{1/p}\left\|f\right\|_{L(p,1)}. \tag{3.22}$$

To prove the second part of the theorem, note that from the above, we have

$$\left\|C_g\right\| = \sup_{\|f\|_{L(p,1)}\leq 1} \frac{\left\|C_g f\right\|_{L(p,1)}}{\left\|f\right\|_{L(p,1)}} \leq C^{1/p}. \tag{3.23}$$

But $\inf\{C : \mu(g^{-1}(A)) \leq C\mu(A)\} = \|g\|$. Thus, $\|C_g\| \leq \|g\|^{1/p}$. To obtain the other inequality, let $f = (1/[\mu(A)]^{1/p})\ \chi_A$. This gives $\|f\|_{L(p,1)} = 1$ and

$$\left\|C_g f\right\|_{L(p,1)} = \left\{\frac{\mu(g^{-1}(A))}{\mu(A)}\right\}^{1/p}. \tag{3.24}$$

Thus,

$$\left\|C_g\right\| = \sup_{\|f\|_{L(p,1)}\leq 1}\left\|C_g f\right\|_{L(p,1)} \geq \sup_{\mu(A)\neq 0}\left\{\frac{\mu(g^{-1}(A))}{\mu(A)}\right\}^{1/p} = \|g\|^{1/p}. \tag{3.25}$$

Now to show the result for $L(p,q)$, note that the operator C_g is linear on $L(p,q)$ and that for p_0 and p_1 such that $p_0 < p < p_1$, we have $\|C_g\chi_A\|_{L(p_0,1)} \leq M_0(\mu(A))^{1/p_0}$ and $\|C_g\chi_A\|_{L(p_1,1)} \leq M_1(\mu(A))^{1/p_1}$. Since $L(p_i,1) \subseteq L(p_i,\infty)$, $i = 0,1$, then for some absolute constants C_0 and C_1 we have

$$\begin{aligned} \|C_g\chi_A\|_{L(p_0,\infty)} &\leq C_0(\mu(A))^{1/p_0}, \\ \|C_g\chi_A\|_{L(p_1,\infty)} &\leq C_0(\mu(A))^{1/p_1}. \end{aligned} \tag{3.26}$$

Hence, by the interpolation theorem we conclude that there is a constant $C > 0$ such that

$$\|C_g f\|_{L(p,q)} \leq C\|f\|_{L(p,q)} \quad \text{for } p_0 < p < p_1,\ \forall q \text{ and } \forall f \in L(p,q). \tag{3.27}$$

□

Remark 3.8. The necessary and sufficient condition (3.14) makes intuitive sense if we consider a variety of measures. Let us consider two of them.

(1) If μ is the Lebesgue measure and X happens to be an interval, then it suffices to take g as the left multiplication by an absolute constant a to achieve (3.14).

(2) If instead μ is the Haar measure, by taking $X = (0,\infty)$, the locally compact topological group of nonzero real numbers with multiplication as operation, then for any Borel set $A \subseteq X$, we have $\mu(A) = \int_A |t|^{-1}dt$. Hence (3.14) is achieved for a measurable function g such that $g^{-1}(A) \subseteq A$. The left multiplication by the reciprocal of an absolute constant a would be enough.

Remark 3.9. The results in Theorems 3.6 and 3.7 are in accordance with the results of Arora et al. in [5, 6]. In fact, even though they obtained their results in a more general version of Lorentz spaces, their necessary and sufficient conditions for boundedness of the multiplication and composition operators are the same as ours.

4. Discussion

The space $L(p,1)$, $p > 1$, seems to be underutilized in analysis despite the fact that in the 1950s Stein and Weiss [9] showed that for a sublinear operator T and a Banach space X if $\|T\chi_A\|_X \leq c\mu(A)^{1/p}$, then $\|Tf\|_X \leq c\|f\|_{L(p,1)}$; that is, T can be extended to the whole $L(p,1)$. De Souza [7] showed that the reason for this is the nature of $L(p,1)$ in that $f \in L(p,1)$ if and only if $f(t) = \sum_{n=0}^{\infty} c_n\chi_{A_n}(t)$ with $\sum_{n=0}^{\infty} |c_n|\mu(A_n)^{1/p} < \infty$. This "atomic decomposition of $L(p,1)$" provides us with a technique to study operators on $L(p,q)$ and in particular L_p. In other words, to study operators on $L(p,q)$, all we need is to study the actions of the operator on characteristic functions which can then be lifted to $L(p,q)$ through the use of interpolation theorems. Although we only considered multiplication and composition operators, other operators (Hardy-Littlewood maximal, Carleson maximal, Hankel, etc.) can be studies likewise.

To conclude, the goal of the present paper is to show that a simple atomic decomposition of $L(p,1)$ spaces shows the boundedness of operators on $L(p,q)$ straightforward.

In fact, we showed that unlike other techniques in the literature, the boundedness of these operators on characteristic functions is enough to generalize to the whole Lorentz space. This technique is not new at all, since it was first used by Stein and Weiss in [9] to extend the Marcinkiewicz interpolation theorem. The broader question is if the the same technique can be extended to Lorentz-Bochner spaces and even Lorentz-martingale spaces. If answered positively, the technique proposed in our paper will contrast the ones by Yong et al. in [10], which we believe are not as straightforward as ours. It has been shown in the literature that operators such as the centered Hardy operator, the Hilbert operator (under Δ_2 condition) are all bounded on Lorentz spaces. Usually the proofs of these facts are not trivial, so by first finding an atomic decomposition on Lorentz spaces $L(p,q),\ 1<p,\ q<\infty$, it would be easier to get another proof of the boundedness of these operators, without having to resort to the Bennett and Sharpley inequality in [11]. It is important to note that, atomic decomposition on general Banach spaces has been found in [12], under the same line of research as ours. Because characteristic functions are easy to manipulate, an even broader question would be to find the class of Banach spaces whose atomic decomposition can be expressed in terms of characteristic functions only.

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