On the equivalence between Weak BMO and the space of derivatives of the Zygmund class

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Abstract

In this paper, we will discuss the space of functions of weak bounded mean oscillation. In particular , we will show that this space is the dual space of the special atom space, whose dual space was already known to be the space of derivative of functions (in the sense of distribution) belonging to the Zygmund class of functions. We show in particular that this proves that the Hardy space H^1 strictly contains the space special atom space.

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1 Introduction

The space of functions of bounded means oscillation has taken central stage in the mathematical literature after the work of Charles Fefferman [4] where he showed that it is the dual space of the real Hardy space $\mathbb{H}^1$, a long sought after result. Right after, Ronald Coifman ([2]) showed this result using using a different method. Essentially, he showed that $\mathbb{H}^1$ has an atomic decomposition. De Souza [3] showed there is a subset B^1 of $\mathbb{H}^1$, formed by special atoms that is contained in $\mathbb{H}^1$. This space B^1 has the particularity that it contains some functions whose Fourier series diverge. The question of whether B^1 is equivalent to $\mathbb{H}^1$ was never truly answered explicitely but it was alway suspected that the inclusion $B^1 \subset \mathbb{H}^1$ was strict, that is, there must be at least one function in $\mathbb{H}^1$ that is not in B^1. However, such a function had never being constructed nor given. Since the dual space $(\mathbb{H}^1)^*$ of $\mathbb{H}^1$ is BMO and $B^1 \subset \mathbb{H}^1$, it follows that the dual space $(B^1)^*$ of B^1 must be a superset of BMO. A natural superset candidate of BMO is therefore the space BMO^w since $BMO \subset BMO^w$. So essence, that BMO^w is the dual of B^1 would also prove that $B^1 \subset \mathbb{H}^1$ with a strict inclusion. Moreover, it was already proved that $(B^1)^* \cong \Lambda_*'$, the space of derivative (in the sense of distributions) of functions in the Zygmund class Λ_*, see for example [3] and [9]. Therefore, if $(B^1)^* \cong BMO^w$, then by transition we would have $\Lambda_*' \cong BMO^w$.

Henceforth, we will adopt the following notations: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit sphere. For an integrable function f on a set A, and the Lebesgue measure λ on A, we will write $\fint_A f(\xi)d\lambda(\xi) := \frac{1}{\lambda(A)}\int_A f(\xi)d\lambda(\xi)$. We will start by recalling the necessary definitions and important results. The interested reader can see for example [5] for more information.

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Definition 1. *Let $0 < p < \infty$ be a real number. The Hardy Space $\mathbb{H}^p := \mathbb{H}^p(\mathbb{D})$ is the space of holomorphic functions f defined on dual and satisfying*

$$\|f\|_{\mathbb{H}^p} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\xi}\right|^p d\lambda(\xi)\right)^{\frac{1}{p}} < \infty\ .$$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n), Q$ be a hypercube in $\mathbb{R}^n$, and λ be the Lebesgue measure on $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Put

$$f_Q^{\#} = \fint_Q f(\xi)d\lambda(\xi), \quad f_Q = \left|\fint_Q f(\xi)d\lambda(\xi)\right|\ .$$

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define

$$M^{\#}(f)(x) = \sup_{Q\ni x}\fint_Q \left|f(\xi) - f_Q^{\#}\right| d\lambda(\xi) \tag{1.1}$$

$$M(f)(x) = \sup_{Q\ni x}\left|\fint_Q [f(\xi) - f_Q]d\lambda(\xi)\right| \tag{1.2}$$

$$mf(x) = \sup_{Q\ni x} f_Q\ . \tag{1.3}$$

where the supremum is taken over all hypercubes Q containing x. Now we can define the space of functions of bounded mean oscillation and its weak counterpart.

Definition 2. *The space of functions of bounded mean oscillation is defined as the space of locally integrable functions f for which the operator $M^{\#}$ is bounded, that is,*

$$BMO(\mathbb{R}^n) = \left\{f \in L^1_{loc}(\mathbb{R}^n) : M^{\#}(f) \in L^\infty(\mathbb{R}^n)\right\}\ .$$

We can endowed $BMO(\mathbb{R}^n)$ with the norm

$$\|f\|_{BMO(\mathbb{R}^n)} = \left\|M^{\#}(f)\right\|_\infty = M^{\#}f(x)\ .$$

The space of functions of weak bounded mean oscillation is defined as the space of locally integrable functions f for which the operator M is bounded, that is,

$$BMO^w(\mathbb{R}^n) = \left\{f \in L^1_{loc}(\mathbb{R}^n) : Mf \in L^\infty(\mathbb{R}^n)\right\}\ .$$

Remark 1.1. *It follows from the above definitions that $BMO(\mathbb{R}^n) \subseteq BMO^w(\mathbb{R}^n)$.*

Let us recall the definition of the space of functions of vanishing mean oscillation $VMO(\mathbb{R}^n)$ and introduce the space of functions of weak vanishing mean oscillations $VMO^w(\mathbb{R}^n)$.

Definition 3. *Let $f \in L^1_{loc}(\mathbb{R}^n)$.*

$$f \in VMO(R^n) \text{ if } \quad \lim_{\lambda(Q)\to 0}\fint_Q |f(\xi) - f_Q|\, d\lambda(\xi) = 0\ .$$

$$f \in VMO^w(\mathbb{R}^n) \text{ if } \quad \lim_{\lambda(Q)\to 0}\left|\fint_Q [f(\xi) - f_Q]d\lambda(\xi)\right| = 0\ .$$

Remark 1.2. *It follows from the above definition that $VMO(\mathbb{R}^n)$ is subspace of $VMO^w(\mathbb{R}^n)$ which is itself a subspace of $BMO^w(\mathbb{R}^n)$. We will show (see Theorem 2.8 below) that VMO^w is in fact closed subspace of BMO^w.*

Henceforth, $BMO(\mathbb{R}^n)$, $BMO^w(\mathbb{R}^n)$, and $VMO^w(\mathbb{R}^n)$ will simply be referred to as BMO, BMO^w, and VMO^w.

Now we consider $A(\mathbb{D})$ as the space of analytic functions defined on the unit disk $\mathbb{D}$. Following the work of Girela in [6] on the space of analytic functions of bounded means oscillations, we introduce their weak counterpart. Before, we recall that the Poisson Kernel is defined as $P_r(\theta) = \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$.

Definition 4. *The space of analytic functions of bounded mean oscillation is defined as*

$$BMOA(\mathbb{D}) = \left\{ F \in A(\mathbb{D}); \exists f \in BMO(\mathbb{T}) : F(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi) f(e^{i\xi}) d\lambda(\xi) \right\} .$$

We can endowed $BMO(\mathbb{D})$ with the norm

$$\|F\|_{BMOA(\mathbb{D})} := |F(0)| + \sup_{\substack{0\le r<1 \\ \theta\in\mathbb{T}}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi) \left| f(e^{i\xi}) - F(re^{i\theta}) \right| d\lambda(\xi) \right) < \infty .$$

In other words, $BMOA(\mathbb{D})$ is the space of Poisson integrals of functions in $BMO(\mathbb{T})$.

We can now define the space $BMOA^w$ of analytic function of weak bounded mean oscillation

Definition 5. *An analytic function F on $\mathbb{D}$ is said to be of weak bounded mean oscillation if there exists $f \in BMO^w(\mathbb{T})$ such that*

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi) f(e^{i\xi}) d\lambda(\xi) .$$

We endow $BMOA^w(\mathbb{D})$ with the norm

$$\|F\|_{BMOA^w(\mathbb{D})} := |F(0)| + \sup_{\substack{0\le r<1 \\ \theta\in\mathbb{T}}} \left(\frac{1}{2\pi} \left| \int_{\mathbb{T}} P_r(\theta-\xi) \left[f(e^{i\xi}) - F(re^{i\theta}) d\lambda(\xi) \right] \right| \right) < \infty .$$

We recall the definition of special atom space B^1, see [8] .

Definition 6. *For $n \ge 1$, we consider the hypercube of $\mathbb{R}^n$ given as $J = \prod_{j=1}^{n} [a_j - h_j, a_j + h_j]$ where a_j, h_j are real numbers with $h_j > 0$. Let $\lambda \in L^1(J)$ with $lambda(J) = \int_J \lambda(\xi) d\xi$*
The special atom (of **type 1***) is a function $b : I \subseteq J \to \mathbb{R}$ such that*

$$\begin{aligned} b(\xi) &= 1 \text{ on } J\backslash I \text{ or} \\ b(\xi) &= \frac{1}{\lambda(I)} \left[\chi_R(\xi) - \chi_L(\xi) \right], \end{aligned}$$

where

- $R = \bigcup_{j=1}^{2^{n-1}} I_{i_j}$ *for some* $i_1, i_2, \cdots, i_{2^{n-1}} \in \{1, 1, \cdots, 2^n\}$ *with* $i_1 < i_2 < \cdots < i_{2^{n-1}}$ *and* $L = I\backslash R$.
- $\{I_1, I_2, \cdots, I_{2^n}\}$ *is the collection of sub-cubes of I, cut by the hyperplanes* $x_1 = a_1, x_2, \cdots, x_n = a_n$.

- χ_A *represents the characteristic function of set* A.

Definition 7. *The special atom space is defined as*

$$B^1 = \left\{ f : J \to \mathbb{R}; f(\xi) = \sum_{n=0}^{\infty} \alpha_n b_n(\xi); \sum_{n=0}^{\infty} |\alpha_n| < \infty \right\} ,$$

where the b_n*'s are special atoms of* **type 1**.

B^1 *is endowed with the norm* $\|f\|_{B^1} = \inf \sum_{n=0}^{\infty} |\alpha_n| ,$ *where the infimum is taken over all representations of* f .

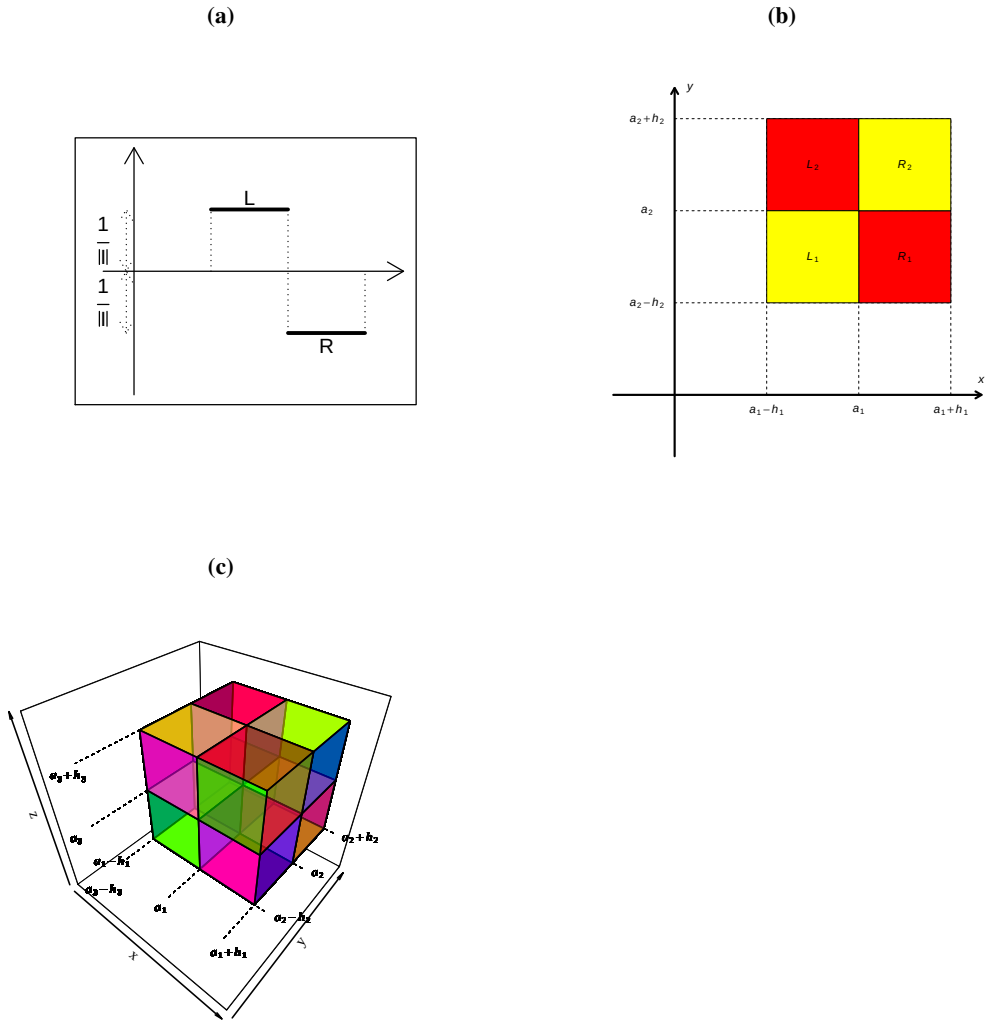

Figure 1: Illustration of the the special atom, for $n = 1$ in **(a)** , $n = 2$ in **(b)**, and $n = 3$ in **(c)**.

Now we define the Zygmund class of functions.

Definition 8. *Let $k \in \mathbb{N}$. A function f is said to be in the Zygmund class $\Lambda_*^k(\mathbb{R}^n)$ of functions of order k if $f \in C^{k-1}(\mathbb{R}^n)$ and*

$$\|f\|_{\Lambda_*^k(\mathbb{R}^n)} = \sum_{|\alpha|=k} \sup_{x,h} \frac{|\partial^\alpha f(x+h) + \partial^\alpha f(x-h) - 2\partial^\alpha f(x)|}{|h|} < \infty .$$

In particular, for $k = 1$, we have $\Lambda_* := \Lambda_*^1(\mathbb{R}^n)$ and defined as

$$\Lambda_* = \left\{ f \in C^0(\mathbb{R}^n) : \|f\|_{\Lambda_*} := \sup_{x,h>0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{2h} < \infty \right\} .$$

One important note about the space Λ_* is that it contains the so-called Weierstrass functions that are known to be continuous everywhere but nowhere differentiable. Therefore, the space Λ_*^k is the space of derivatives of functions in Λ_*^{k-1}, where the derivative is taken in the sense of distributions. Another equivalent way to see Λ_*^k is to consider functions of Λ_*^{k-1} that are either differentiable or limits of convolutions with the Poisson kernel, that is, $f(\xi) = \lim_{r\to 1}(f * P_r)(\xi)$ where $P_r(\theta)$ is the Poisson kernel.

2 Main Results

Our first result is about the constant f_Q in the definition of BMO^w. If fact, the constant f_Q can be replaced with any non-negative constant. In the case of BMO, f_Q can be replaced by any constant.

Proposition 2.1. *Let $f \in L^1_{loc}$.*

(1) For any non-negative real number α, we have

$$\sup_{Q\ni x} \left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| \le 2 \sup_{Q\ni x} \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| . \tag{2.1}$$

(2) $f \in BMO^w$ if and only if for any $x \in \mathbb{R}^n$ and any cube $Q \ni x$, there exists is a non-negative number $\alpha \in Q$ such that

$$\sup_{Q\ni x} \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| < \infty. \tag{2.2}$$

Proof. (1) To prove the first part, suppose $x \in \mathbb{R}^n$ and let a cube $Q \subseteq \mathbb{R}^n$ containing x. We have

$$\left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| \le 2 \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| .$$

Indeed, for any non-negative real number α, we have

$$\begin{aligned}
\left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| &= \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) + \fint_Q [\alpha - f_Q] d\lambda(\xi) \right| \\
&\le \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| + \left| \fint_Q [\alpha - f_Q] d\lambda(\xi) \right| \\
&\le \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| + \left| \left| \fint_Q \alpha d\lambda(\xi) \right| - \left| \fint_Q f(\xi) d\lambda(\xi) \right| \right| \\
&\le \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| + \left| \fint_Q [\alpha - f(\xi)] d\lambda(\xi) \right| = 2 \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| .
\end{aligned}$$

To conclude, we taking the supremum over of all cubes $Q \ni x$.
(2)"$\Longrightarrow$" Let $f \in BMO^w$, then

$$\sup_{Q \ni x} \left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| < \infty \, .$$

Thus (2.2) is satisfied with $\alpha = \inf \ f_Q$.

"$\Longleftarrow$" Suppose there exists is a non-negative number α such that $\sup_{Q \ni x} \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| < \infty$, then from equation (2.1) above, $\sup_{Q \ni x} \left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| \leq 2 \sup_{Q \ni x} \left| \fint_Q [f(\xi) - \alpha] d\lambda(\xi) \right| < \infty$, that is, $f \in BMO^w$.

□

We can define two equivalent norms on BMO^w and prove that endowed with these norms, BMO^w is in fact a Banach space.

Proposition 2.2. *Consider the following:*

$$\|f\|_{BMO^w} = \|mf\|_\infty + \|Mf\|_\infty \quad \textit{and} \quad \|f\|^*_{BMO^w} = \|mf\|_\infty + 2 \sup_{Q \ni x} \inf_{\alpha > 0} \left| \fint_Q [f(\xi) - f_Q] d\lambda(\xi) \right| .$$

Then

$$\|f\|_{BMO^w} \leq \|f\|^*_{BMO^w} \leq 2 \|f\|_{BMO^w} \, .$$

Proof. The proof is an immediate consequence of Proposition 2.1. We note that the proof can also be obtained from the Closed-graph theorem, but that will require to first prove that endowed with the two norms, BMO^w is a Banach space. □

Theorem 2.3. *The following is true:* $(BMO^w, \|\cdot\|_{BMO^w})$ *is Banach space.*

Proof. (1). In the proof that $\|\cdot\|_{BMO^w}$ is a norm, homogeneity and the triangle inequality are easy to prove. As for positivity, we note that $\|f\|_{BMO^w} = 0 \Longleftrightarrow \sup_{Q \ni x} \ f_Q = 0$ and $f(\xi) = f_Q$ on all cubes $Q \ni x$. It follows immediately that $f = f_Q = 0$.

(2). Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in BMO^w. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\forall n, m \in N$, we have $\|f_n - f_m\|_{BMO^w} < \epsilon$. That is,

$$\sup_{Q \ni x} \left[(f_n - f_m)_Q + \left| \fint_Q [(f_n - f_m)(\xi) - (f_n - f_m)_Q] d\lambda(\xi) \right| \right] < \epsilon \, . \tag{2.3}$$

In particular from equation (2.3), we have that $\sup_{Q \ni x} \ (f_n - f_m)_Q < \epsilon$, therefore

$$\begin{aligned} |f_{n,Q} - f_{m,Q}| &= \left| \left| \fint_Q f_n(\xi) d\lambda(\xi) \right| - \left| \fint_Q f_m(\xi) d\lambda(\xi) \right| \right| \\ &\leq \left| \fint_Q [f_n - f_m](\xi) d\lambda(\xi) \right| = (f_n - f_m)_Q \\ &\leq \sup_{Q \ni x} (f_n - f_m)_Q < \epsilon \end{aligned}$$

Hence for fixed Q, $\{f_{n,Q}\}$ is Cauchy sequence in $\mathbb{R}$, a complete space. Let $f_Q = \lim_{n\to\infty} f_{n,Q}$. We note from the above inequalities that

$$(f_n - f_m)_Q = \left| \fint_Q [f_n - f_m](\xi) d\lambda(\xi) \right| \geq |f_{n,Q} - f_{m,Q}| \geq f_{n,Q} - f_{m,Q} \ .$$

Therefore, given a cube $x \in \mathbb{R}^n$

$$\begin{aligned} |Mf_n(x) - Mf_m(x)| &\leq \sup_{Q \ni x} \left| \fint_Q [(f_n(\xi) - f_m(\xi)) - (f_{n,Q} - f_{m,Q})] d\lambda(\xi) \right| \\ &\leq \sup_{Q \ni x} \left| \fint_Q [(f_n(\xi) - f_m(\xi)) - (f_n - f_m)_Q] d\lambda(\xi) \right| + \sup_{Q \ni x} |(f_n - f_m)_Q - (f_{n,Q} - f_{m,Q})| \\ &\leq \|f_n - f_m\|_{BMO^w} + \sup_{Q \ni x} |(f_{n,Q} - f_{m,Q})| < 2\epsilon \ . \end{aligned}$$

It follows that $\{Mf_n\}_{n\in\mathbb{N}}$ is a Cauchy-sequence in L^∞_{loc}, a Banach space. Let $h = \lim_{n\to\infty} Mf_n$. Since $L^\infty_{loc} \subset L^1_{loc}$, then we also have $h \in L^1_{loc}$.

$$\begin{aligned} h(x) &= \lim_{n\to\infty} Mf_n(x) \\ &= \lim_{n\to\infty} \sup_{Q \ni x} \left| \fint_Q [f_n(\xi) - f_{n,Q}] d\lambda(\xi) \right| \\ &= \sup_{Q \ni x} \left| \fint_Q [\lim_{n\to\infty} f_n(\xi) - f_Q] d\lambda(\xi) \right| \ . \end{aligned}$$

Since $h(x)$ is finite on any cube $Q \ni x$, that follows that $f(\xi) = \lim_{n\to\infty} f_n(\xi)$ is finite a.e. on Q. Thus $h = Mf$, for some $f \in BMO^w$ and $\lim_{n\to\infty} \|f_n - f\|_{BMO^w} \to 0$. □

Theorem 2.4 (Hölder's type inequality)**.** *Let $g \in BMO^w$ and a hyper-cube $J \subset \mathbb{R}^n$. Consider the following operator defined on B^1: $T_g : B^1 \to \mathbb{R}$ with $T_g(f) = \int_J f(\xi)g(\xi)d\lambda(\xi)$. Then*

1. *$T_g \in (B^1)^*$ with $\|T_g\|_{(B^1)^*} \leq \|g\|_{BMO^w}$.*
2. *Moreover, the operator $H : BMO^w \to (B^1)^*$ defined as $H(g) = T_g$ is onto.*

Proof. Let $g \in BMO^w$ and $T_g : B^1 \to \mathbb{R}$ with $T_g(f) = \int_J f(\xi)g(\xi)d\lambda(\xi)$. By linearity of the integral, T_g is a linear operator on B^1. To start, we study the action of this operator on special atoms. Indeed, let $\xi \in I \subseteq J$ and suppose $f(\xi) = b(\xi) = \frac{1}{\lambda(I)}[\chi_R(\xi) - \chi_L(\xi)]$, where R, L are sub-intervals of I such that $I = R \cup L$ and $R \cap L = \emptyset$. Therefore we have:

$$\begin{aligned} T_g(b) &= \int_J b(\xi)g(\xi)d\lambda(\xi) = \int_I b(\xi)g(\xi)d\lambda(\xi) \\ &= \int_I b(\xi)[g(\xi) - g_I]d\lambda(\xi), \quad \text{since } \int_I b(\xi)g_I d\lambda(\xi) = 0 \ . \end{aligned}$$

Taking the supremum over all $I \ni x$ and using the fact that $b(\xi) \leq \frac{1}{\lambda(I)}$, we have

$$|T_g(b)| \leq \sup_{\substack{I \ni x \\ I \subseteq J}} \inf_{\alpha > 0} \left| \fint_I [g(\xi) - \alpha] d\lambda(\xi) \right| \leq \|g\|_{BMO^w} \ . \tag{2.4}$$

Now suppose $f(\xi) = \sum_{n=1}^{\infty} c_n b_n(\xi)$ with $\sum_{n=1}^{\infty} |c_n| < \infty$ is an element of B^1, where the b_n's are special atoms defined on sub-intervals I_n of J with $I_n = R_n \cup L_n$ and $R_n \cap L_n = \emptyset$. Let $I = \bigcup_{n=1}^{\infty} I_n$. We have for $\alpha > 0$

$$\begin{aligned} T_g(f) &= \int_J \left(\sum_{n=1}^{\infty} c_n b_n(\xi) \right) g(\xi) d\lambda(\xi) \\ &= \int_I \left(\sum_{n=1}^{\infty} c_n b_n(\xi) \right) [g(\xi) - \alpha] \, d\lambda(\xi) \\ &= \sum_{n=1}^{\infty} c_n \int_I b_n(\xi) \, [g(\xi) - \alpha] \, d\lambda(\xi) \\ &= \sum_{n=1}^{\infty} c_n \int_{I_n} b_n(\xi) \, [g(\xi) - \alpha] \, d\lambda(\xi) \, . \end{aligned}$$

It follows from (2.4) that

$$|T_g(f)| \leq \left(\sum_{n=1}^{\infty} |c_n| \right) \cdot \|g\|_{BMO^w} \, .$$

Talking the infimum over all representations of f yields:

$$|T_g(f)| \leq \|f\|_{B^1} \cdot \|g\|_{BMO^w} \, .$$

Therefore T_g is a bounded linear operator on B^1 with

$$\|T_g\|_{(B^1)^*} = \sup_{\|f\|_{B^1}=1} |T_g(f)| \leq \|g\|_{BMO^w} \, . \tag{2.5}$$

Now suppose that T is a bounded linear functional on B^1, that is, $T \in (B^1)^*$. We want to show that there exists a function $g \in BMO^w$ such that $T(f) = T_g(f) = \int_J f(\xi) g(\xi) d\lambda(\xi)$. That $T \in (B^1)^*$ implies the existence of an absolute constant C such that

$$|T(f)| \leq C \, \|f\|_{B^1} \, , \quad \forall f \in B^1. \tag{2.6}$$

Recall that a function $G : J \to \mathbb{R}$ is said to be absolutely continuous on J if for every positive number ϵ, there exists a positive number δ, such that for a finite sequence of pairwise disjoint sub-intervals $I_n = (x_n, y_n)$ of J,

$$\sum_{n=1}^{\infty} (y_n - x_n) < \delta \implies \sum_{n=1}^{\infty} |G(y_n) - G(x_n)| < \epsilon \, . \tag{2.7}$$

Suppose such a sequence of sub-intervals exists. Now consider $G(x) = T\left(\chi_{[x-h, x+h)}\right)$ for some real number $h > 0$. Then, given an interval I_n and a real number h_n, we have by linearity of T that

$$G(y_n) - G(x_n) = T\left(\left[\chi_{[y_n - h_n, y_n + h_n)} - \chi_{[x_n - h_n, x_n + h_n)} \right] \right) \, .$$

Now if we define $L_n = [x_n - h_n, x_n + h_n)$ and $R_n = [y_n - h_n, y_n + h_n)$, we see that $R_n \cap L_n = \emptyset$ their union forms a single interval J_n if we choose $h_n = (y_n - x_n)/2$. Moreover, in that case, the length of the interval J_n is $y_n + h_n - x_n + h_n = 2(y_n - x_n)$. Therefore by linearity

$$G(y_n) - G(x_n) = 2(y_n - x_n) T\left(\frac{1}{2(y_n - x_n)} \left[\chi_{[y_n - h_n, y_n + h_n)} - \chi_{[x_n - h_n, x_n + h_n)} \right] \right) \, .$$

We observe that $b_n(\xi) = \dfrac{1}{2(y_n - x_n)}\left[\chi_{[y_n - h_n, y_n + h_n)}(\xi) - \chi_{[x_n - h_n, x_n + h_n)}(\xi)\right]$ is a special atom with norn in B^1 equal to 1. It follows that using (2.6)

$$|G(y_n) - G(x_n)| \leq 2(y_n - x_n)\,|T(b_n)| \leq 2C(y_n - x_n)\,\|b_n\|_{B^1} = 2C(y_n - x_n)\,.$$

Hence,

$$\sum_{n=1}^{\infty} |G(y_n) - G(x_n)| \leq 2C \sum_{n=1}^{\infty} (y_n - x_n)\,.$$

We conclude by noting that given $\epsilon > 0$, (2.7) is satisfied if we choose $\delta = \dfrac{\epsilon}{2C}$. We conclude that the function G is absolutely continuous on J. Therefore, G is differentiable almost everywhere, that is, there exists $g \in L^1$ such that $G(x) = \displaystyle\int_a^x g(\xi)d\lambda(\xi)$ for all intervals $I = [a, b] \subseteq J$. Let $I \ni x$ be a sub-interval of J. Therefore, $\sup\limits_{I \ni x} \dfrac{1}{\lambda(I)} \left| \displaystyle\int_I g(\xi)d\lambda(\xi) \right| < \infty$ because an absolutely continuous function is a function with bounded variation. Since $g \in L^1$, we have that $g_I = \dfrac{1}{\lambda(I)} \left| \displaystyle\int_I g(\xi)d\lambda(\xi) \right| < \infty$. It follows that

$$Mg(x) = \sup_{I \ni x} \frac{1}{\lambda(I)} \left| \int_I (g(\xi) - g_I)d\lambda(\xi) \right| \leq \sup_{I \ni x} \frac{1}{\lambda(I)} \left| \int_I g(\xi)d\lambda(\xi) \right| + \sup_{I \ni x}\ g_I < \infty\,.$$

This proves that $g \in BMO^w$. We have therefore shown that for every bounded linear functional T on B^1, there exists $g \in BMO^w$ such that $T(f) = T_g(f) = \displaystyle\int_J f(\xi)g(\xi)d\lambda(\xi)$. That is, $H : BMO^w \to (B^1)^*$ is onto with $H(g) = T = T_g$. Identifying T with the g, it follows from (2.5) that

$$\|g\|_{(B^1)^*} = \|T\|_{(B^1)^*} = \|T_g\|_{(B^1)^*} \leq \|g\|_{BMO^w}\ . \tag{2.8}$$

□

Remark 2.5. *We observe that the above result can be obtained differently. In fact, we recall that it was proved in [3] that the dual space of B^1 is equivalent to Λ^2_*. Let $x \in J, h > 0, I = [x - h, x + h] \subseteq J, L = [x - h, x)$, and $R = [x, x + h]$. Let $b(\xi) = \dfrac{1}{2h}[\chi_R(\xi) - \chi_L(\xi)]$. We have*

$$\begin{aligned} \|f\|_{(B^1)^*} \cong \|f\|_{\Lambda^2_*} &= \sup_{\substack{I \ni x \\ h > 0}} \left| \int_I b(\xi)\partial f(\xi)d\lambda(\xi) \right| \\ &= \sup_{\substack{I \ni x \\ h > 0}} \left| \int_I b(\xi)(\partial f(\xi) - \partial f_I)d\lambda(\xi) \right| \\ &\leq \|f\|_{BMO^w}\ . \end{aligned}$$

This shows that $BMO^w \subseteq \Lambda^2_* \cong (B^1)^*$ and the inequality in (2.8) follows.

Theorem 2.6. *The dual space $(B^1)^*$ of B^1 is BMO^w with $\|g\|_{BMO^w} \cong \|g\|_{(B^1)^*}$.*

To prove Theorem 2.6, we recall that $B^1(\mathbb{T})$ has an analytic extension $B^1_A(\mathbb{D})$ to the unit disk. In fact, in [1], it was shown that $B^1_A(\mathbb{D})$ consists of functions F that are Poisson integrals of functions in $B^1(\mathbb{T})$, that is, $F(z) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \frac{1 + e^{-i\xi}z}{1 - e^{-i\xi}z} f(e^{i\xi})d\lambda(\xi)$ where $f \in B^1(\mathbb{T})$. Moreover, the norm

$\|F\|_{B^1_A(\mathbb{D})} = \int_0^1 \int_{-\pi}^{\pi} \left|F'(z)\right| dz$ is equivalent to the norm $\|f\|_{B^1(\mathbb{T})}$. This allows to identify $B^1_A(\mathbb{D})$ with $B^1(\mathbb{T})$ and thus the following:

Proposition 2.7. *$B^1_A(\mathbb{D})$ can be identified with a closed subspace of $L^1(\mathbb{T})$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B^1(\mathbb{T})$ that converges to f in $L^1(\mathbb{T})$. We need to show that the Poisson integral of f is in $B^1_A(\mathbb{D})$. Since $f_n \in B^1(\mathbb{T}), \forall\ n \in \mathbb{N}$, then $F_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{-i\xi}z}{1-e^{-i\xi}z} f_n(e^{i\xi}) d\lambda(\xi)$ belongs to $B^1_A(\mathbb{D})$. Let $F(z)$ be the Poisson integral of f. We note that if $z = e^{i\theta}$, then $(\xi-\theta)^2 - \frac{(\xi-\theta)^4}{2} \leq \left|1-e^{-i\xi}z\right|^2 \leq (\xi-\theta)^2$. Therefore, we have that $F'(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2e^{-i\xi}}{(1-e^{-i\xi}z)^2} f(e^{i\xi}) d\lambda(\xi)$ and $\left|F'(z)\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2}{|1-e^{-i\xi}z|^2}\left|f(e^{i\xi})\right| d\lambda(\xi) \leq C\,\|f\|_{L^1(\mathbb{T})}$. It follows that ,

$$\|F_n - F\|_{B^1_A} = \int_0^1 \int_{-\pi}^{\pi} \left|F_n'(z) - F'(z)\right| dz \leq C\,\|f_n - f\|_{L^1(\mathbb{T})}\,.$$

Since $f_n \to f$ in $L^1(\mathbb{T})$, it follows that $F \in B^1_A$ and $F_n \to F$ in $B^1_A(\mathbb{D})$. The result follows by identifying $B^1_A(\mathbb{D})$ and $B^1(\mathbb{T})$. □

We note that there is an extension of this result to the polydisk $\mathbb{D}^n$ and polysphere $\mathbb{T}^n$, see [7] .

Proof of theorem 2.6 . It sufficient to show that there exists a constant $M > 0$ such that $\|T\|_{(B^1(\mathbb{T}))^*} \geq M\,\|g\|_{BMO^w(\mathbb{T})}$. The extension to $\mathbb{R}^n$ is natural, using the results in [7, 8] . With Proposition 2.7, the proof follows along the lines of the proof of Proposition 7.3 in [6].
Let $T \in (B^1(\mathbb{T}))^*$. Since $B^1(\mathbb{T})$ is a closed subspace of $L^1(\mathbb{T})$ by Proposition 2.7, then by Hahn-Banach Theorem, T can be extended to a bounded linear operator $T' \in (L^1(\mathbb{T}))^*$ with $\|T\|_{(B^1(\mathbb{T}))^*} = \|T'\|_{(L^1(\mathbb{T}))^*}$. Since $(L^1(\mathbb{T}))^* = L^\infty(\mathbb{T})$, then there exists $g_0 \in L^\infty(\mathbb{T})$ such that $\|g\|_{L^\infty(\mathbb{T})} = \|T'\|_{(L^1)^*} = \|T\|_{(B^1)^*}$ and $T(f) = \frac{1}{2\pi}\int_{\mathbb{T}} \overline{g_0(e^{i\xi})} f(e^{i\xi}) d\lambda(\xi)$, for all $f \in B^1_A(\mathbb{D})$. Note that here, we identify f with its correspondent in $B^1_A(\mathbb{D})$. Now let $\sum_{n\in\mathbb{N}} A_n e^{in\xi}$ be the Fourier series of g_0. Since $g_0 \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$, we have that $\sum_{n\in\mathbb{Z}} |A_n|^2 < \infty$. This means that g_0 is holomorphic. Let

$$g(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{g_0(e^{i\xi})}{e^{i\xi}-z} d(e^{i\xi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g_0(e^{i\xi})}{1-e^{-i\xi}z} d\xi\,.$$

Since $\frac{1}{1-e^{i\xi}z} = \sum_{n\in\mathbb{N}} e^{-in\xi} z^n$ and $A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\xi} g_0(e^{i\xi}) d\lambda(\xi)$, we have

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g_0(e^{i\xi})}{1-e^{-i\xi}z} d\xi = \sum_{n\in\mathbb{N}} A_n z^n\,.$$

This implies that $g \in H^2(\mathbb{D})$. For $\theta \in \mathbb{R}$, $\overline{g(e^{i\theta})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\overline{g_0(e^{i\xi})}}{1-e^{-i\xi}e^{i\theta}} d\lambda(\xi)$. Moreover, given $f \in B^1_A(\mathbb{D}) \subseteq H^1(\mathbb{D})$, and using the Cauchy integral formula, we have:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{g(e^{i\theta})} f(e^{i\theta}) d\theta \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{g_0(e^{i\theta})} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\xi})}{1-e^{-i\xi}e^{i\theta}} d\lambda(\xi)\right) d\theta$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g_0(e^{i\theta})}f(e^{i\theta})d\theta \\
&= T(f)
\end{aligned}
$$

We also observe that

$$
g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left[\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}+1\right]g_0(e^{i\xi})d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_0(e^{i\xi})d\lambda(\xi), \tag{2.9}
$$

where $G_0(e^{i\xi}) = \frac{1}{2}\left(g_0(e^{i\xi}) + \frac{1}{2\pi}\int_{-\pi}^{\pi}g_0(e^{i\xi})d\lambda(\xi)\right) = \frac{1}{2}\left(g_0(e^{i\xi}) + A_0\right)$.
Put $G_1 = \text{Re}(G_0), G_2 = \text{Im}(G_0)$, and

$$
U(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_1(e^{i\xi})d\lambda(\xi); V(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_2(e^{i\xi}) .
$$

Then $g = U + iV$. Moreover, U and V are analytic in $\mathbb{D}$ since they represent the Poisson integral of $G_1, G_2 \in L^\infty(\mathbb{T}) \subseteq B^1(\mathbb{T})$. Observe that $BMO \subseteq BMO^w$. Moreover, Theorem 3.2 in [6] shows that $\|g\|_{BMO} \cong \|g\|_{BMOA}$. It therefore follows that there exists a constant $C > 0$ such that

$$
C\,\|g\|_{BMO^w} \leq C\,\|g\|_{BMO} \leq \|g\|_{BMOA} \leq \left\|T'\right\|_{(L^1)^*} = \|T\|_{(B^1)^*} .
$$

□

2.1 Discussion

We note that $BMO^w \subseteq \Lambda_*'$ with $\|f\|_{\Lambda_*'} \leq C\,\|f\|_{BMO^w}$ where $\|g\|_{\Lambda_*'} = \|f\|_{\Lambda_*}$ with $g' = f$ in the sense of distributions. Since $(B^1)^* \cong \Lambda_*'$, and from Theorem 2.6 above $(B^1)^* \cong BMO^w$, it follows that $BMO^w \cong \Lambda_*'$. The consequence is that there exists $c > 0$ such that $c\,\|f\|_{BMO^w} \leq \|f\|_{\Lambda_*'}$, that is, those two norms are equivalent. We finish by noting Λ_*' has an analytic characterization, so we would expect the analytic characterization of BMO^w to also be equivalent to that of Λ_*'.
Another takeaway from Theorem 2.6 is that it provides another proof that BMO is strictly contained in BMO^w otherwise we would have had $B^1 \cong \mathbb{H}^1$, which is not true. In other words, there exists $f \in \mathbb{H}^1$ such that $f \notin B^1$.
Our next result is about the closeness of VMO^w in BMO^w.

Theorem 2.8. *VMO^w is a closed subspace of BMO^w.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in VMO^w that converges to $f \in BMO^w$. Let us prove that $f \in VMO^w$. That $f_n \to f$ as $n \to \infty$ in BMO^w is equivalent to $\lim_{n\to\infty}\|f_n - f\|_{BMO^w} = 0$. The latter is also equivalent, by definition of the norm in BMO^w to

$$
\lim_{n\to\infty}\sup_{Q\ni x}(f_n - f)_Q = 0 \text{ and } \quad \lim_{n\to\infty} M(f_n - f)(x) = 0 .
$$

Since $f \in BMO^w$, then $\sup_{Q\ni x}\left|⨍_Q [f(\xi) - f_Q]d\lambda(\xi)\right| < \infty$. Therefore

$$
\begin{aligned}
\left|⨍_Q [f(\xi) - f_Q]d\lambda(\xi)\right| &= \left|⨍_Q [f(\xi) - f_n(\xi) + f_n(\xi) - f_{n,Q} + f_{n,Q} - f_Q]d\lambda(\xi)\right| \\
&\leq \left|⨍_Q [f(\xi) - f_n(\xi)]d\lambda(\xi)\right| + \left|⨍_Q [f_n(\xi) - f_{n,Q}]d\lambda(\xi)\right| + \left|⨍_Q [f_{n,Q} - f_Q]d\lambda(\xi)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left| \fint_Q [f(\xi) - f_n(\xi)] d\lambda(\xi) \right| + \left| \fint_Q [f_n(\xi) - f_{n,Q}] d\lambda(\xi) \right| + |f_{n,Q} - f_Q| \\
&\leq \left| \fint_Q [f(\xi) - f_n(\xi)] d\lambda(\xi) \right| + \|f_n - f\|_{BMO^w} \ . \\
&\leq \|f_n - f\|_{L^1_{loc}} + \|f_n - f\|_{BMO^w} \ .
\end{aligned}
$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of function is BMO^w, we have that $\lim_{n\to\infty} \|f_n - f\|_{BMO^w} = 0$. Since $BMO^w \subset L^1_{loc}$, we have that $\lim_{n\to\infty} \|f_n - f\|_{L^1_{loc}} = 0$. Hence VMO^w is a closed subspace of BMO^w. □

Remark 2.9. *We note that it was proved in [6] that VMO is a closed subspace of BMO. A stronger result even showed that if we restrict ourselves to $\mathbb{T}$, then the space of complex-valued and continuous functions $C(\mathbb{T}) \subseteq VMO(\mathbb{T})$ and the $BMO(\mathbb{T})$-closure of $C(\mathbb{T})$ is precisely $VMO(\mathbb{T})$. It turns out this result is also true for VMO^w.*

Theorem 2.10. *The BMO^w-closure of $C(\mathbb{T})$ is precisely $VMO^w(\mathbb{T})$, that is*

$$
\overline{C(\mathbb{T})}^{BMO^w} = VMO^w(\mathbb{T}) \ .
$$

Proof. We observe that $C(\mathbb{T}) \subseteq VMO(\mathbb{T}) \subseteq VMO^w(\mathbb{T}) \subseteq BMO^w(\mathbb{T})$. Therefore since $VMO^w(\mathbb{T})$ is closed in $BMO^w(\mathbb{T})$, we have that

$$
\overline{C(\mathbb{T})}^{BMO^w(\mathbb{T})} \subseteq \overline{VMO^w(\mathbb{T})}^{BMO^w(\mathbb{T})} = VMO^w(\mathbb{T}) \ .
$$

For the other direction , let $f \in VMO^w(\mathbb{T})$, and consider the sequence $\{R_\epsilon(f)\}_{\epsilon>0}$ of rotations of f by angle ϵ, defined on $\mathbb{T}$ as $R_\epsilon(f)(e^{i\theta}) = f(e^{i(\theta-\epsilon)}); \theta \in \mathbb{R}$. Then from Theorem 2.1 in [6], we have that for all $\epsilon > 0, R_\epsilon(f) \in C(\mathbb{T})$ and $\lim_{\epsilon\to 0} \|R_\epsilon(f) - f\|_{BMO(\mathbb{T})} = 0$. Since $BMO(\mathbb{T}) \subseteq BMO^w(\mathbb{T})$, we also have that $\lim_{\epsilon\to 0} \|R_\epsilon(f) - f\|_{BMO^w(\mathbb{T})} = 0$. That is, $f \in \overline{C(\mathbb{T})}^{BMO^w(\mathbb{T})}$.

□

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