Counting process

Introduction

A counting process is a nonnegative, integer-valued, increasing stochastic process. The most common use of a counting process is to count the number of occurrences of some event of interest as time goes by, and the index set is therefore usually taken to be the nonnegative real numbers $[0, \infty)$ (although the more general index set $R = (-\infty, \infty)$ is also commonly used). Formally, a counting process $\{N(t), t \geq 0\}$ is then any nonnegative, integer-valued stochastic process such that $N(s) \leq N(t)$ whenever $s \leq t$. For the purpose of this article, we shall assume that the count starts at 0 so that $N(0) \equiv 0$ (this assumption can be relaxed).

A sequence of random variables that arises naturally in a counting process is the "`sojourn`" time" T_k between the kth and $(k+1)$th event for $k \geq 0$, and the quantity $N(t)$ can be expressed in terms of T_k by using the standard technique of "`indicator functions`:" the indicator function I_A of an event A equals 1 if the event occurs and 0 otherwise. Letting $S_k = T_0 + T_1 + \cdots T_k$, the time of the kth event, it holds that

$$N(t) = \sum_{k=0}^{\infty} I_{\{S_k \leq t\}} \tag{1}$$

and the number of events in a given time interval $(s, t]$ can be defined as $N(s, t) := N(t) - N(s)$. Most commonly, assumptions are made regarding the distribution of the sojourn times, both with respect to the distribution of individual times and the dependence structure between times. Depending on the assumptions, a counting process may be known under other names and fit into other general classes of processes. One such assumption is that the sojourn times are independent with T_k following an exponential distribution with mean μ_k leading to a "`pure birth process`," which can be used to model a population of reproducing individuals where an "event" is the birth of a new individual and the population size increases by one. This in turn is a special case of a "`birth and death process`" (where population size may also decrease) and is the only case in which a counting process is a "**Markov process**." Another assumption is that the sojourn times $T_1, T_2, \ldots$ are independent and identically distributed (i.i.d.) in which case the resulting process is known as a "`renewal process`," which is a thoroughly studied and much applied type of process. If, in addition, the T_k have an exponential distribution with mean μ (so that we have a process that is both a pure birth process and a renewal process), the result is a homogeneous "**Poisson process**" with "`rate`" $\lambda := 1/\mu$ (the mean number of events per time unit). Renewal processes and the Poisson process are described in more detail subsequently.

Renewal Processes

Recall that $N(t)$ is the number of events up to time t. A fundamental question is how $N(t)$ behaves as t increases and one answer is provided by a simple application of the "`Law of Large Numbers`." With $\mu = E[T_k]$, the mean sojourn time, it can be shown that

$$\frac{N(t)}{t} \longrightarrow \frac{1}{\mu}$$

as $t \to \infty$ (the convergence holds "`almost surely`," that is, with probability 1). The expected value $m(t) = E[N(t)]$, is called the "`renewal function`" and perhaps surprisingly, it is significantly more difficult to establish the asymptotic behavior of $m(t)$. However, the result that finally follows is equally as intuitive as the previous one:

$$\frac{m(t)}{t} \longrightarrow \frac{1}{\mu}$$

and thus, both $N(t)$ and $m(t)$ increase roughly as μt as t increases. It is also worth noticing that by (1), $m(t)$ can be given explicitly in terms of the "`convolution powers`" of the distribution a sojourn time:

$$m(t) = \sum_{k=0}^{\infty} F^{*k}(t)$$

where the convolution F^{*k} is the cumulative **distribution function** (cdf) of S_k (and F^{*0} is interpreted as a unit point mass at 0).

One particular complication in applications of renewal processes is that observation of the process often starts at some time when the process itself may

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already have started. In other words, the distribution of T_1 is not necessarily the same as that of the other T_k. If observation starts at time t, there have been an unknown number $N(t)$ of previous events, the next event will be numbered $N(t) + 1$, and our first observation occurs in the random interval $[T_{N(t)}, T_{N(t)+1})$. The "`excess lifetime`" at time t is defined as

$$W(t) = T_{N(t)+1} - t$$

and is thus the time (previously denoted T_1) to the first event from the time observation starts ("W" for "wait"). The distribution of $W(t)$ generally depends on t and is thus not the same as that of the T_k; in fact, it is only so if the T_k are exponentially distributed. The following asymptotic result holds for the distribution of $W(t)$:

$$P(W(t) \leq x) \longrightarrow \frac{1}{\mu} \int_0^x (1 - F(y))\mathrm{d}y \qquad (2)$$

where F is the common cdf of the T_k (some technical restrictions on F are needed, for example, that it has a density with respect to Lebesgue measure). The limiting distribution in (2) can be used in applications as the (approximate) distribution of T_1. It may be tempting to believe that $E[W(t)]$ must be smaller than the $E[T_k]$ because time has already elapsed since the previous event but this is not true; in the other direction works the fact that an arbitrarily chosen starting time is more likely to follow in an interval that is longer than average (simply because the longer the interval, the easier it is to "hit"). In other words, the distribution of $T_{N(t)+1}-, T_{N(t)}$ is in general not the same as that of the T_k (again, this is only so in the exponential case). This phenomenon is known as the "**waiting time**," which will be addressed below.

The Poisson Process

In a Poisson process with rate λ, it is a standard result that the number of events $N(s, t)$ follows a "**Poisson distribution**" with mean $\lambda(t - s)$ and that $N(s, t)$ and $N(u, v)$ are independent for all disjoint time intervals $(s, t]$ and $(u, v]$. This consequence is in fact equivalent to the aforementioned assumption of exponential sojourn times and thus provides an alternative characterization of the Poisson process. As the Poisson process is a Markov process, it has the memory-less property: at any time t, the time until the next event follows an exponential distribution with mean μ regardless of when the most recent event before t occurred (and anything that happened before it). In the Poisson process, the excess lifetime $W(t)$ thus has the same exponential distribution as the T_k (and it is readily checked that this exponential distribution arises in (2) above). Now note that at any time t, the time since the *previous* event also follows an exponential distribution with mean μ. Thus, the time between the previous and next event has mean 2μ, which seems to contradict the fact that the time between any two consecutive events has mean μ. This is the waiting time paradox whose explanation was given above; the interval in which we start does not have the typical sojourn time distribution; in fact, it was just argued that in the Poisson process it is on average twice as long as the typical sojourn time.

Further Reading

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(*See also* **Distribution function**; **Markov chains**; **Markov process**; **Poisson process**; **Waiting time**; **Poisson distribution**)

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