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GENERAL BRANCHING PROCESSES WITH IMMIGRATION

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Abstract

A general multi-type branching process where new individuals immigrate according to some point process is considered. An intrinsic submartingale is defined and a convergence result for processes counted with random characteristics is obtained. Some examples are given.

GENERAL BRANCHING PROCESS; MULTI-TYPE; IMMIGRATION

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1. Introduction

One extension of branching process theory is to allow *immigration* into the population. This is certainly natural from a biological or demographical point of view and such processes have been studied for some decades now. In a recent paper by Lyons *et al.* (1995) a new application of immigration appears. They present a new and very elegant proof of the classical Kesten–Stigum theorem, that the martingale limit in a supercritical Galton–Watson process is non-degenerate if and only if the reproduction law has a finite $x \log x$ moment. The proof is based on comparing different measures on the set of trees and it turns out that a certain *Galton–Watson process with immigration* is central. The basic theorem for such processes is the following (see Asmussen and Hering (1983)).

Theorem 1.1. *Consider a Galton–Watson process where the offspring law has mean* $m > 1$. *Assume that* Y_n *individuals immigrate in the nth generation and let* Z_n *be the size of the nth generation. Let*

$$S = \sum_{k=1}^{\infty} \frac{Y_k}{m^k}.$$

Then $\lim_{n\to\infty} Z_n/m^n$ *exists and is finite a.s. on the set* $\{S < \infty\}$. *If in addition the offspring law has a finite* $x \log x$ *moment, then* $Z_n/m^n \to \infty$ *on* $\{S = \infty\}$.

Note that no assumptions are made on the sequence $\{Y_n\}$. The i.i.d. case was settled by Seneta (1970) who showed that $S < \infty$ if and only if $E[\log^+ Y_1] < \infty$. It is also shown that if $E[\log^+ Y_1] = \infty$, then there is no way to normalise Z_n to obtain a finite non-degenerate limit.

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Generalisations to Bellman–Harris processes and Markov branching processes appear in Asmussen and Hering (1976). General branching processes with immigration have not attracted much interest; a result for single-type processes under second moment conditions for both reproduction and immigration may be found in Jagers (1975).

This paper uses ideas and methods from Asmussen and Hering (1976) to prove immigration results for general multi-type branching processes. In order to make it reasonably self-contained, we give a short description of such processes in the next section, following Jagers (1992).

2. General branching processes

Individuals are identified by descent; the ancestor is denoted by 0, the individual $x = (x_1, \cdots, x_n)$ belongs to the nth generation and is the x_nth child of the x_{n-1}th child of...of the x_1th child of the ancestor. This gives the *Ulam–Harris space*

$$I = \bigcup_{n=0}^{\infty} N^n,$$

the set of all individuals.

When an individual is born she is given a *type* picked from the *type space* $(S, \mathscr{S})$ where $\mathscr{S}$ is countably generated. The type $s \in S$ determines a probability measure $P(s, \cdot)$, the *life kernel* on the *life space*, denoted by Ω and equipped with a countably generated σ-algebra $\mathscr{A}$. The information provided by a life, i.e. an element $\omega \in \Omega$, may differ from one application to another. It is always required though, that we can define the *reproduction process* ξ on $S \times R_+$, giving the sequence of types $\sigma(1), \sigma(2), \cdots$ of children and their mothers' ages $\tau(1), \tau(2), \cdots$ at their births, so that

$$\xi(A \times [0, t]) = \#\{k : \sigma(k) \in A, 0 \leqq \tau(k) \leqq t\}.$$

We put $\tau(k) = \infty$ if fewer than k children are born.

With each individual x, we associate its type σ_x, its life ω_x, its *birth-time* τ_x and its *reproduction process* ξ_x. The ancestor is born at time $\tau_0 = 0$, other individuals' birth-times being defined inductively through their mothers' ages at child-bearing.

The set of life kernels $\{P(r, \cdot), r \in S\}$ together with a starting condition that the ancestor be of type s, defines a probability measure P_s on $(S \times \Omega^I, \mathscr{S} \times \mathscr{A}^I)$, the *population space*. The measure P_s is such that each individual x initiates a new branching process which obeys the law P_{σ_x} and, given σ_x, is conditionally independent of everything outside x's progeny.

An important entity is the *reproduction kernel*, μ, defined through

$$\mu(s, dr \times dt) = E_s[\xi(dr \times dt)],$$

the expectation of ξ when the mother is of type $s \in S$. The process is assumed to be *strictly Malthusian* and *supercritical*, meaning that there is a number $\alpha > 0$ such that the kernel

$$\hat{\mu}_\alpha(s, dr) = \int_0^\infty e^{-\alpha t}\mu(s, dr \times dt)$$

has Perron root 1 and is *conservative* in the sense of Shurenkov (1992). Then there exists a measure π and a function h such that

$$\int_S \hat{\mu}_\alpha(s, dr)\pi(ds) = \pi(dr),$$

$$\int_S h(r)\hat{\mu}_\alpha(s, dr) = h(s).$$

Further we assume that $\inf_s h(s) > 0$, $h \in L^1[\pi]$ and that the *stable age of childbearing*, β, defined through

$$\beta = \int_{S \times S \times R_+} t\,e^{-\alpha t}h(r)\mu(s, dr \times dt)\pi(ds)$$

is finite. Then π can be normed to a probability measure and we use the notation E_π for expectation when the type is distributed according to π, i.e. $E_\pi = \int_S E_s\pi(ds)$.

Finally, we assume that the reproduction kernel is non-lattice (for a strict definition of this concept in the multi-type setting, see Shurenkov (1992)) and satisfies the condition

$$\sup_s \mu(s, S \times [0, a]) < 1,$$

for some $a > 0$.

To count, or measure, the population *random characteristics* are used. A random characteristic is a real-valued process χ, where $\chi(a)$ gives the contribution of an individual of age a. We assume that χ is non-negative and vanishing for negative a (no individual contributes before her birth). The χ-value pertaining to the individual x is denoted by χ_x and the *χ-counted population*, z_t^χ is defined as

$$z_t^\chi = \sum_{x \in I} \chi_x(t - \tau_x),$$

i.e. the sum of the contributions of all individuals (at time t the invidual x is of age $t - \tau_x$). The simplest example of a random characteristic is $\chi(t) = I_{R_+}(t)$, which is zero before you are born and one afterwards. Then z_t^χ is the total number of individuals born up to time t, denoted by y_t.

Under the conditions on μ above and a direct Riemann integrability assumption on the function $E_s[e^{-\alpha t}\chi(t)]$, the general Markov renewal theorem of Shurenkov (1992) applies and gives the asymptotics of $E_s[e^{-\alpha t}z_t^\chi]$, the expected χ-counted population when the ancestor's type is s. The convergence result is the following.

Theorem 2.1. *Consider a strictly Malthusian, non-lattice branching process, counted with a bounded characteristic χ such that the function $E_s[e^{-\alpha t}\chi(t)]$ is directly Riemann integrable. Then, for π-almost all $s \in S$,*

$$E_s[\mathrm{e}^{-\alpha t} z_t^\chi] \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} h(s),$$

as $t \to \infty$.

When dealing with the asymptotics of the process $\mathrm{e}^{-\alpha t} z_t^\chi$ itself, the *intrinsic martingale*, w_t, introduced (for single-type processes) by Nerman (1981), is central. See also Jagers (1989). Denote x's mother by mx and let

$$\mathscr{I}_t = \{x : \tau_{mx} \leqq t < \tau_x\},$$

the set of individuals whose mothers are born before time t, but who themselves are born after t. Now let

$$w_t = \sum_{x \in \mathscr{I}_t} \mathrm{e}^{-\alpha \tau_x} h(\sigma_x),$$

the individuals in $\mathscr{I}_t$ summed with time- and type-dependent weights. If $\mathscr{F}_t$ is the σ-algebra generated by the types and lives of all individuals born before time t, then $\{w_t, \mathscr{F}_t\}$ is a real-time non-negative martingale with $E_s[w_t] = h(s)$. Hence there exists a limit w such that $w_t \to w$ a.s. as $t \to \infty$. Uniform integrability of w_t is guaranteed by the $x \log x$ condition, that, in terms of

$$\bar{\xi} = \int_{S \times R_+} \mathrm{e}^{-\alpha t} h(s) \xi(ds \times dt),$$

$E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty$. Under this condition, $w_t \to w$ in $L^1[P_s]$ for π-almost all $s \in S$ and the following theorem holds.

Theorem 2.2. *Add to the conditions of* Theorem 2.1 *that the* $x \log x$ *condition is satisfied and that, for any* $t \geqq 0$, $\sup_s E_s[y_t; y_t > c] \to 0$ *as* $c \to \infty$. *Then*

$$\mathrm{e}^{-\alpha t} z_t^\chi \to \frac{E[\hat{\chi}(\alpha)]}{\alpha\beta} w,$$

in $L^1[P_s]$ *for* π*-almost all* $s \in S$ *as* $t \to \infty$.

For a proof, see Jagers (1989).

3. Processes with immigration

Now consider a general branching process where new individuals $v(1), v(2), \cdots$ immigrate according to some point process η with points of occurrence $(\sigma_{v(1)}, \tau_{v(1)}, \cdots)$, where the kth immigrant initiates a branching process according to the law $P_{\sigma_{v(k)}}$. With Γ denoting the set of realisations of point processes on $S \times R_+$ equipped with an appropriate σ-algebra $\mathscr{G}$ and a probability measure Q giving the distribution of η, the *immigration space* is $(\Gamma, \mathscr{G}, Q)$. Further, with $\mathbf{\Omega} = \Omega^I$, the *population space* is $(\Gamma \times \mathbf{\Omega}^\infty, \mathscr{G} \times \mathscr{A}^\infty, P)$, where P is the joint law of the immigration process and the population. Note that it is not assumed that the processes initiated by consecutive immigrants are independent or even conditionally independent given the sequence of the immigrants' types. If such conditional independence is assumed then, with $G \in \mathscr{G}$, $A_k \in \mathscr{A}$, $k = 1, 2, \cdots$ and $(\sigma_k(\gamma), \tau_k(\gamma))$ denoting the kth point in $\gamma \in \Gamma$,

$$P\left(G \times \prod_{k=1}^{\infty} A_k\right) = \int_G \prod_{k=1}^{\infty} P_{\sigma_k(\gamma)}(A_k) Q(d\gamma).$$

Let the process start with no individuals at time 0 and let $z_t^\chi(k)$ denote the process started by $v(k)$. The resulting process, denoted by $\hat{z}_t^\chi$, can then be described as the superposition

$$\hat{z}_t^\chi = \sum_{k=1}^{\eta(t)} z_{t-\tau_{v(k)}}^\chi(k), \tag{3.1}$$

a χ-measured general branching process with immigration. Note that since $z_t^\chi = 0$ for negative t, the sum may just as well run from one to infinity. This will be done now and then in the sequel. A useful fact is that with the special characteristic

$$\chi^*(t) = \mathrm{e}^{\alpha t} \int_S \int_t^\infty \mathrm{e}^{-\alpha u} h(r) \xi(dr \times du),$$

we have $w_t = \mathrm{e}^{-\alpha t} z_t^{\chi^*}$. If we use the notation $\hat{w}_t$ when there is immigration and w_t when there is not, and let $\mathscr{Y}$ denote the σ-algebra generated by $(\sigma_{v(1)}, \tau_{v(1)}, \cdots)$, we obtain, by (3.1),

$$\hat{w}_t = \sum_{k=1}^{\eta(t)} \exp[-\alpha \tau_{v(k)}] w_{t-\tau_{v(k)}}(k). \tag{3.2}$$

Since η has points of occurrence $(\sigma_{v(1)}, \tau_{v(1)}, \cdots)$,

$$\bar{\eta} = \int_{S \times R_+} \mathrm{e}^{-\alpha t} h(s) \eta(ds \times dt) = \sum_{k=1}^{\infty} \exp[-\alpha \tau_{v(k)}] h(\sigma_{v(k)})$$

and the following lemma holds.

Lemma 3.1. *Under the conditional probability measure* $P(\cdot \mid \mathscr{Y})$, $\{\hat{w}_t, \mathscr{F}_t\}$ *is a non-negative submartingale that has bounded expectations on the event* $\{\bar{\eta} < \infty\}$.

Proof. The submartingale property is immediate since the individuals in $\mathscr{I}_{s+t}$ either stem from individuals in $\mathscr{I}_t$ or from immigrants in the time interval $(t, t+s]$. Hence, by the martingale property of w_t,

$$E[\hat{w}_{t+s} \mid \mathscr{F}_t, \mathscr{Y}] = \hat{w}_t + \sum_{k=\eta(t)}^{\eta(t+s)} E[\chi^*_{v(k)}(t+s-\tau_{v(k)}) \mid \mathscr{Y}] \geqq \hat{w}_t.$$

Further, by (3.2) and the fact that $E_s[w_t] = h(s)$,

$$E[\hat{w}_t \mid \mathcal{Y}] = E\left[\sum_{k=1}^{\eta(t)} \exp[-\alpha\tau_{\nu(k)}]w_{t-\tau_{\nu(k)}}(k) \mid \mathcal{Y}\right]$$
$$= \sum_{k=1}^{\eta(t)} \exp[-\alpha\tau_{\nu(k)}]E[w_{t-\tau_{\nu(k)}}(k) \mid \mathcal{Y}]$$
$$= \sum_{k=1}^{\eta(t)} \exp[-\alpha\tau_{\nu(k)}]h(\sigma_{\nu(k)}) \to \bar{\eta},$$

as $t \to \infty$, so that $\sup_t E[\hat{w}_t \mid \mathcal{Y}] < \infty$ on $\{\bar{\eta} < \infty\}$.

Hence there exists a random variable $\hat{w} < \infty$ such that $\hat{w}_t \to \hat{w}$, $t \to \infty$ on $\{\bar{\eta} < \infty\}$, all of this holding a.s. with respect to $P(\cdot \mid \mathcal{Y})$. Since $P(A) = E[P(A \mid \mathcal{Y})]$ we obtain the following result.

Corollary 3.2. *On $\{\bar{\eta} < \infty\}$ there exists an a.s. finite random variable $\hat{w}_\infty$ such that $\hat{w}_t \to \hat{w}_\infty$ a.s. as $t \to \infty$.*

Now recall (3.2). Since all the $w_{t-\tau_{\nu(k)}}(k)$ converge to their respective martingale limits $w(k)$, the natural question is whether these limits can be put in the sum to obtain a limit for $\hat{w}_t$. Therefore define

$$\hat{w} = \sum_{k=1}^{\infty} \exp[-\alpha\tau_{\nu(k)}]w(k). \tag{3.3}$$

The fundamental result is the following theorem.

Theorem 3.3. *Consider a supercritical, strictly Malthusian branching process with immigration process η such that $\bar{\eta} < \infty$ a.s. With $\hat{w}$ as above, $\hat{w}_t \to \hat{w}$, a.s. as $t \to \infty$.*

Proof. Since $\hat{w}_t \to \hat{w}_\infty$ we must prove that $\hat{w}_\infty = \hat{w}$ a.s. on $\{\bar{\eta} < \infty\}$. By (3.1) and (3.3), for $s \leqq t$,

$$\hat{w}_\infty - \hat{w} = (\hat{w}_\infty - \hat{w}_t) + \sum_{\tau_{\nu(k)} \leqq s} \exp[-\alpha\tau_{\nu(k)}](w_{t-\tau_{\nu(k)}}(k) - w(k))$$
$$+ \sum_{s < \tau_{\nu(k)} \leqq t} \exp[-\alpha\tau_{\nu(k)}]w_{t-\tau_{\nu(k)}}(k) + \sum_{\tau_{\nu(k)} \geqq s} \exp[-\alpha\tau_{\nu(k)}]w(k),$$

where the first and second term tends to 0 as $t \to \infty$. The last term tends to 0 as $s \to \infty$ since $\bar{\eta} < \infty$. For the third term, note that for fixed s it converges to a finite limit and that this limit is decreasing in s and hence has an a.s. non-negative limit as $s \to \infty$. Hence

$$E\left[\liminf_{s\to\infty} \liminf_{t\to\infty} \sum_{s < \tau_{\nu(k)} \leqq t} \exp[-\alpha\tau_{\nu(k)}]w_{t-\tau_{\nu(k)}}(k) \mid \mathcal{Y}\right]$$
$$\leqq \liminf_{s\to\infty} \liminf_{t\to\infty} \sum_{s < \tau_{\nu(k)} \leqq t} \exp[-\alpha\tau_{\nu(k)}]E[w_{t-\tau_{\nu(k)}}(k) \mid \mathcal{Y}]$$
$$= \liminf_{s\to\infty} \liminf_{t\to\infty} \sum_{s < \tau_{\nu(k)} \leqq t} \exp[-\alpha\tau_{\nu(k)}]h(\sigma_{\nu(k)}),$$

which equals 0 since $\bar{\eta} < \infty$ a.s.

4. Process convergence

Under the $x \log x$ condition, $\{w_t, t \geqq 0\}$ is uniformly integrable and hence it converges in $L^1(P_s)$ for π-almost all $s \in S$. However, this does not imply that $\{\hat{w}_t, t \geqq 0\}$ is uniformly integrable without an extra condition on the immigration process, as is seen from the following theorem.

Theorem 4.1. *Assume that* $E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty$. *Then* $\{\hat{w}_t, t \geqq 0\}$ *is uniformly integrable if and only if* $E[\bar{\eta}] < \infty$.

Proof. Since $E[\hat{w}_t \mid \mathcal{Y}] \to \bar{\eta}$ it follows by monotone convergence that $E[\hat{w}_t] \to E[\bar{\eta}]$. Under the $x \log x$ condition, $E_s[w] = h(s)$ and hence

$$E[\hat{w} \mid \mathcal{Y}] = \sum_{k=1}^{\infty} \exp[-\alpha\tau_{\nu(k)}] E[w(k) \mid \mathcal{Y}] = \sum_{k=1}^{\infty} \exp[-\alpha\tau_{\nu(k)}] h(\sigma_{\nu(k)}) = \bar{\eta},$$

so that $E[\hat{w}] = E[\bar{\eta}]$. Since $\hat{w}_t \to \hat{w}$ a.s. it follows that the convergence holds also in L^1 if $E[\bar{\eta}] < \infty$ (see Durrett (1991), p. 224). Since $\hat{w}_t$ is a submartingale this is equivalent to uniform integrability. If $E[\bar{\eta}] = \infty$, $E[\hat{w}_t]$ is unbounded and hence $\hat{w}_t$ cannot be uniformly integrable.

For the asymptotics of the χ-counted population, further restrictions on Q are needed because of the π-a.e. qualification in Theorem 2.1. Clearly we should require the set of immigration processes which contain 'bad' starting types to be a Q-null set. Therefore, let S_0 be the set of types for which the convergence in Theorem 2.2 does not hold and let

$$G_0 = \{\gamma \in \Gamma : \sigma_k(\gamma) \in S_0 \text{ for some } k = 1, 2, \cdots\}.$$

The following theorem holds.

Theorem 4.2. *Consider a general branching process satisfying the conditons of* Theorem 2.2. *With the immigration process denoted by* η *and* G_0 *as above, assume that* $Q(G_0) = 0$, *and that* $E[\bar{\eta}] < \infty$. *Then*

$$e^{-\alpha t} \hat{z}_t^{\chi} \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} \hat{w},$$

in L^1 *as* $t \to \infty$. *Conversely, if* $E[\bar{\eta}] = \infty$ *then* $E[e^{-\alpha\tau} \hat{z}_t^{\chi}] \to \infty$.

Proof. Let $\rho = E_\pi[\hat{\chi}(\alpha)]/\alpha\beta$. By (3.1) and (3.3),

$$E[|e^{-\alpha t} \hat{z}_t^{\chi} - \rho\hat{w}|] \leqq E\left[\sum_{k=1}^{\eta(t)} |\exp[-\alpha\tau_{\nu(k)}] \exp[-\alpha(t - \tau_{\nu(k)})] z_{t-\tau_{\nu(k)}}^{\chi}(k) - \rho \exp[-\alpha\tau_{\nu(k)}] w(k)|\right]$$

$$+ \rho E\left[\sum_{k=\eta(t)+1}^{\infty} \exp[-\alpha\tau_{\nu(k)}] w(k)\right],$$

where the second term tends to 0 since $E[\hat{w}] = E[\bar{\eta}] < \infty$ and the first term equals

$$\int_\Gamma \sum_{k=1}^{\gamma(t)} \exp[-\alpha\tau_k(\gamma)] E_{\sigma_k(\gamma)}[|\exp[-\alpha(t-\tau_k(\gamma))] z^\chi_{t-\tau_k(\gamma)}(k) - \rho w(k)|] Q(d\gamma).$$

From the proof of Theorem 3 in Jagers (1992) we know that there exists a constant C such that $E_s[\mathrm{e}^{-\alpha t} z_t^\chi] \leqq Ch(s)$ for all s and t. Since for any $s \notin S_0$,

$$E_s[|\mathrm{e}^{-\alpha t} z_t^\chi - \rho w|] \to 0$$

and

$$\int_\Gamma \sum_{k=1}^{\infty} \exp[-\alpha\tau_k(\gamma)] h(\sigma_k(\gamma)) Q(d\gamma) = E[\bar{\eta}],$$

which is finite by assumption, dominated convergence applies to obtain

$$E[|\mathrm{e}^{-\alpha t} \hat{z}_t^\chi - \rho \hat{w}|] \to 0,$$

at $t \to \infty$.

Now assume that $E[\bar{\eta}] = \infty$ and note that, by (3.1),

$$E[\mathrm{e}^{-\alpha t} \hat{z}_t^\chi] = \int_\Gamma \sum_{k=1}^{\infty} \exp[-\alpha\tau_k(\gamma)] E_{\sigma_k(\gamma)}[\exp[-\alpha(t-\tau_k(\gamma))] z^\chi_{t-\tau_k(\gamma)}(k)] Q(d\gamma),$$

which goes to infinity as $t \to \infty$ by Theorem 2.1 and Fatou's lemma.

5. Examples

5.1. *The general single-type process.* In this case $\bar{\eta}$ reduces to

$$\bar{\eta} = \int_0^\infty \mathrm{e}^{-\alpha t} \eta(dt),$$

the Laplace transform of η, usually denoted by $\hat{\eta}(\alpha)$. Assuming that immigrants produce *independent* copies of the branching process, a converse to Theorem 3.3 is easily obtained.

Corollary 5.1. *Consider a general single-type branching process with immigration process η such that immigrants produce independent copies of the branching process. Then $\hat{w}_t \to \hat{w}$, a.s. on $\{\hat{\eta}(\alpha) < \infty\}$ as $t \to \infty$. If the $x \log x$ condition holds, then $\hat{w}_t \to \infty$ a.s. on $\{\hat{\eta}(\alpha) = \infty\}$.*

Proof. The first part follows immediately from Theorem 3.3. For the second part, first note that $E[w(k); w(k) \leqq \exp[\alpha\tau_{\nu(k)}] \mid \mathcal{Y}] \geqq \frac{1}{2} E[w(k) \mid \mathcal{Y}]$ for k large enough since the $w(k)$ are now i.i.d. Under the $x \log x$ condition, $E[w(k)] = 1$ so that $\hat{\eta}(\alpha) = E[\hat{w} \mid \mathcal{Y}]$ a.s. Therefore if $\hat{\eta}(\alpha) = \infty$ also

$$\sum_{k=1}^{\infty} \exp[-\alpha\tau_{\nu(k)}] E[w(k); w(k) \leqq \exp[\alpha\tau_{\nu(k)}] \mid \mathcal{Y}] = \infty$$

and $\hat{w} = \infty$ a.s. by Kolmogorov's three series theorem.

A special case of this is the Galton–Watson process for which Theorem 1.1 is easily obtained by considering the immigration process

$$\eta(dt) = \sum_{k=1}^{\infty} \delta_k(dt)\, Y_k .$$

5.2. *Renewal immigration.* Assume that the immigration time-points $\tau_{\nu(1)}, \tau_{\nu(2)}, \cdots$ constitute a renewal process. Then $\tau_{\nu(k)}$ is a sum of k i.i.d. variables with mean μ say so that $\tau_{\nu(k)} \sim k\mu$ as $k \to \infty$. Hence, for any $\varepsilon > 0$, eventually $\exp[-\alpha\tau_{\nu(k)}] \leqq (\exp[-\alpha(\mu-\varepsilon)])^k$ a.s. Under the conditions of Jagers (1992), h is bounded and hence $\bar{\eta} < \infty$. Clearly the case $\mu = \infty$ only makes the situation more favourable.

References

Asmussen, S. and Hering, H. (1976) Strong limit theorems for supercritical immigration-branching processes. *Math. Scand.* **39**, 327–342.

Asmussen, S. and Hering, H. (1983) *Branching Processes.* Birkhäuser, Boston.

Durrett, R. (1991) *Probability: Theory and Examples.* Wadsworth, Pacific Grove, CA.

Jagers, P. (1975) *Branching Processes with Biological Applications.* Wiley, Chichester.

Jagers, P. (1989) General branching processes as Markov fields. *Stoch. Proc. Appl.* **32**, 183–212.

Jagers, P. (1992) Stabilities and instabilities in population dynamics. *J. Appl. Prob.* **29**, 770–780.

Lyons, R., Pemantle, R. and Peres, Y. (1995) Conceptual proofs of $L \log L$ criteria for mean behaviour of branching processes. *Ann. Prob.* **3**, 1125–1138.

Nerman, O. (1981) On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrscheinlichkeitsth.* **57**, 365–395.

Seneta, E. (1970) On the supercritical branching process with immigration. *Math. Biosci.* **7**, 9–14.

Shurenkov, V. M. (1992) Markov renewal theory and its applications to Markov ergodic processes. *Preprint.* Department of Mathematics, Chalmers University of Technology and Göteborg University.