1 Math Stat, Test 1, due March 7, noon

1. Let X and Y be independent observations of a quantity that has mean μ and variance σ^2 where we wish to estimate μ. Consider the two estimators

$$\widehat{\mu} = \frac{2X + Y}{3} \quad \text{and} \quad \widetilde{\mu} = \frac{3X - Y}{2}$$

(a) Show that both estimators are unbiased.
(b) Which estimator is better and why?

2. If the random variable X has pdf

$$f(x) = x^{a-1}(1-x)^{b-1}\frac{(a+b-1)!}{(a-1)!(b-1)!}, \quad 0 \le x \le 1$$

it is said to have a *beta distribution* with parameters a and b (note that the particular choice $a = b = 1$ gives the uniform distribution). Let $X_1, ..., X_n$ be a sample from a beta distribution with $b = 2$ and find the MOME of a. Also compute the value of $\widehat{a}$ if we have the observations $0.3, 0.6, 0.9$.

3. Consider a Poisson process with unknown rate λ where we know that times between consecutive points are i. i. d. and $\exp(\lambda)$. The time S until the kth point has a so-called *gamma distribution*, $S \sim \Gamma(k, \lambda)$, which has pdf

$$f_\lambda(t) = e^{-\lambda t}\frac{\lambda^k\, t^{k-1}}{(k-1)!}, \quad t \ge 0$$

(a) Find the MLE $\widehat{\lambda}$ of λ if we fix k (so that k is known) and the kth point is observed at time T.

Note: We have one observation, that is, the sample size $n = 1$.

(b) If we want to find a confidence interval for λ based on the estimator $\widehat{\lambda}$, we need to use the fact that $S \sim \Gamma(k, \lambda)$ where k is known. Suggest a function of the type $T(\widehat{\lambda}, \lambda)$ that has a known distribution without any unknown parameters, upon which we can base our confidence interval (you don't have to find the interval). Hint: If $X \sim \exp(\lambda)$, how can you transform X to get rid of λ in the distribution?

4. Let $Z_1, Z_2, \ldots$ be independent random variables that are $N(0,1)$. For each of the following random variables, determine whether it has a t or a χ^2 distribution or neither.

(a) $X = \dfrac{Z_1^2}{2} + \dfrac{Z_2^2}{2} - Z_1 Z_2$

(b) $X = \dfrac{\sqrt{2} Z_1}{\sqrt{Z_2^2 + Z_3^2}}$

(c) $X = (Z_1 - Z_2)^2 + (Z_3 - Z_4)^2$

(d) $X = \dfrac{\sqrt{Z_1^2}}{\sqrt{(Z_2^2 + Z_3^2)/2}}$

5. Metal rods are being manufactured and are supposed to have a length of 10 inches. A sample of 30 measured lengths gave sample mean 10.13 and a sample variance of 0.16. Find a 95% confidence interval for the mean length μ. State assumptions. Do you think that the manufacturing process is working as it should?

6. Write a love poem that includes the word "homoscedastic."