1 Math Stat, Solutions to test 1

1(a) By properties of expected vaules:

$$E[\widehat{\mu}] = E\left[\frac{2X+Y}{3}\right] = \frac{1}{3}(2E[X] + E[Y]) = \frac{1}{3}(2\mu + \mu) = \mu$$

so $\widehat{\mu}$ is unbiased. Further

$$E[\tilde{\mu}] = E\left[\frac{3X-Y}{2}\right] = \frac{1}{2}(3E[X] - E[Y]) = \frac{1}{2}(3\mu - \mu) = \mu$$

so $\tilde{\mu}$ is unbiased as well.

(b) The estimator with the smaller variance is better. By properties of variances of independent random variables we get

$$\mathrm{Var}[\widehat{\mu}] = \mathrm{Var}\left[\frac{2X+Y}{3}\right] = \frac{1}{3^2}(2^2 \cdot \mathrm{Var}[X] + \mathrm{Var}[Y]) = \frac{5\sigma^2}{9}$$

and

$$\mathrm{Var}[\tilde{\mu}] = \mathrm{Var}\left[\frac{3X-Y}{2}\right] = \frac{1}{2^2}(3^2 \cdot \mathrm{Var}[X] + (-1)^2 \cdot \mathrm{Var}[Y]) = \frac{10\sigma^2}{4}$$

so since $\widehat{\mu}$ has the smaller variance, it is the better estimator.

2. The first moment is

$$m_1 = E[X] = \int_0^1 x \cdot x^{a-1}(1-x)\frac{(a+1)!}{(a-1)!}dx = \frac{a}{a+2}$$

which we solve for a to get

$$a = \frac{2m_1}{1-m_1}$$

which gives the MOME

$$\widehat{a} = \frac{2\widehat{m}_1}{1-\widehat{m}_1} = \frac{2\bar{X}}{1-\bar{X}}$$

With the observed values, we have $\bar{X} = 0.6$ and get $\widehat{a} = 3$.

3(a) The likelihood function is

$$L(\lambda) = e^{-\lambda T} \, \frac{\lambda^k \, T^{k-1}}{(k-1)!}$$

which has logarithm

$$\ln L(\lambda) = -\lambda S + k \ln \lambda + (k-1) \ln T - \ln(k-1)!$$

and we get

$$\frac{d}{d\lambda} \ln L(\lambda) = -T + \frac{k}{\lambda} = 0$$

which gives the MLE $\widehat{\lambda} = \dfrac{k}{T}$.

(b) If $X \sim \exp(\lambda)$, then $\lambda X \sim \exp(1)$ because

$$P(\lambda X \leq x) = P(X \leq \frac{x}{\lambda}) = 1 - e^{-\lambda \frac{x}{\lambda}} = 1 - e^{-x}$$

Hence, the random variable λS is the sum of n i. i. d. random variables that are $\exp(1)$ so $\lambda S \sim \Gamma(n, 1)$ where there are no unknown parameters. Hence, the function

$$T(\widehat{\lambda}, \lambda) = \frac{\widehat{\lambda}}{\lambda} = \frac{k}{\lambda T}$$

has a distribution that is completely known.

4(a) Since

$$X = \left(\frac{Z_1 - Z_2}{\sqrt{2}} \right)^2$$

and

$$\frac{Z_1 - Z_2}{\sqrt{2}} \sim N(0, 1)$$

because linear combination of normals are normal, $E[Z_1 - Z_2] = 0$, and $\text{Var}[Z_1 - Z_2] = 2$, we conclude that $X \sim \chi^2_1$.

(b) Since

$$X = \frac{Z_1}{\sqrt{\dfrac{Z_2^2 + Z_3^2}{2}}}$$

we conclude that $X \sim t_2$.

(c) Neither. This is the sum of squares of independent normals but they are not $N(0,1)$ since each has variance 2.

(d) Neither. As the numerator equals $|Z_1|$, it is not $N(0,1)$ and although the denominator is of the right form, X does not have a t distribution.

5. The confidence interval is

$$\mu = \bar{X} \pm t\frac{s}{\sqrt{n}} \quad (q)$$

where we get $t = 2.045$ (t-table, DF$= n - 1 = 29, (1 + 0.95)/2 = 0.975$) and hence

$$\mu = 10.13 \pm 2.045\frac{\sqrt{0.16}}{\sqrt{30}} = 10.13 \pm 0.15 \quad (0.95)$$

Since the interval includes 10, the process seems to be working as it should.