Math Stat, solutions to HW2

1(a) The mean in the unif$[\theta, 1]$ is $(\theta+1)/2$ and as the sample mean has the same mean as the distribution, $E[\bar{X}] = (\theta+1)/2$. Thus, $\theta = 2E[\bar{X}] - 1$ which suggests the unbiased estimator $\widehat{\theta} = 2\bar{X} - 1$ (this is the MOME). To show that it really is unbiased, note that

$$E[\widehat{\theta}] = 2E[\bar{X}] - 1 = 2\mu - 1 = 2(\theta+1)/2 - 1 = \theta$$

The variance is

$$\mathrm{Var}[\widehat{\theta}] = 4\mathrm{Var}[\bar{X}] = 4 \cdot \frac{(1-\theta)^2}{12n} = \frac{(1-\theta)^2}{3n}$$

which gives the standard error

$$\sigma_{\widehat{\theta}} = \frac{1-\theta}{\sqrt{3n}}$$

We have $n = 5$, get $\bar{X} = -0.22$ which gives $\widehat{\theta} = 2(-0.22) - 1 = -1.44$. The estimated standard error is

$$\widehat{\sigma_{\widehat{\theta}}} = \frac{1-(-1.44)}{\sqrt{15}} = 0.63$$

(b) The better estimator should be based on the minimum $X_{(1)}$. If we have n observations, on average they are equidistantly spaced over the interval $[\theta, 1]$, dividing it into $n+1$ subintervals, each of length $(1-\theta)/(n+1)$. Thus, the expected value of the minimum should be

$$E[X_{(1)}] = \theta + \frac{1-\theta}{n+1} = \frac{n\theta+1}{n+1}$$

which can be shown explicitly by finding the pdf of $X_{(1)}$ and the usual integration. This observation suggests the unbiased estimator

$$\widehat{\theta} = \frac{(n+1)X_{(1)} - 1}{n}$$

which by analogy with the uniform $[0, \theta]$ example from class ought to be more efficient than the estimator from **(a)** (and this can also be proved strictly by computing its variance). Note that θ can be any number less than 1, including negative numbers.

2. By the variance formula, $\mathrm{Var}[s] = E[s^2] - E[s]^2 = \sigma^2 - E[s]^2$ and since $\mathrm{Var}[s] > 0$ we get $\sigma^2 - E[s]^2 > 0$, that is, $E[s] < \sigma$.