Mathematical Statistics, solutions to test 2

1(a) We have $n = 1166, \widehat{p} = 612/1166 = 0.52$ and with $q = 0.95$ we get $z = 19.6$ and the interval

$$p = 0.52 \pm 1.96\sqrt{\frac{0.52 \cdot 0.48}{1166}} = 0.52 \pm 0.029 \quad (0.95)$$

As the interval includes 0.5, there is not support for the claim.

(b) The margin of error for a 95% confidence interval with sample size n equals

$$1.96\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$$

so if we want this ≤ 0.01 we get the equation

$$1.96\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} = 0.01$$

which gives

$$n = \frac{1.96^2 \widehat{p}(1-\widehat{p})}{0.01^2}$$

The worst-case scenario (maximal margin of error) is if $\widehat{p} = 0.5$ and this gives

$$n = \frac{1.96^2 \cdot 0.25}{0.01^2} = 9604$$

Using the estimate from **(a)** gives

$$n = \frac{1.96^2 \cdot 0.52 \cdot 0.48}{0.01^2} = 9589$$

Note that the answer varies depending on how you round off.

(c) Test $H_0 : p = 0.5$ vs. $H_A : p > 0.5$. Recall that the null hypothesis should specify one parameter value (we could state it as $H_0 : p \leq 0.5$ but need to use $p = 0.5$ in the calculations) and that it is the alternative that you are trying to prove.

2(a) After a vigorous discussion in class, the answer is that it depends on how we construct our confidence intervals to draw the conclusions. If we do a lower-bounded interval of the type $\mu \geq \bar{X} - ts/\sqrt{n}$ and issue a warning when the latter number is greater than c, we should decrease the confidence level to get a shorter interval and thus more warnings. If we instead to an upper-bounded interval of the type $\mu \leq \bar{X} + ts/\sqrt{n}$ and issue a warning when this is greater than c (issue an "all clear" statement when the confidence interval is entirely below c), we should increase the confidence level to get a longer interval and thus more warnings.

(b) In **(a)** we are dealing with a better-safe-than-sorry scenario and don't care if we warn too often as long as we miss a few real cases as possible. The opposite of **(a)** would be any situation in which the consequences of issuing a warning are more dire than failing to issue one, for example a similar situation but one where health risks are not severe but issuing a warning may lead to production shut-down which may be costly and impractical.

3(a) As the probability a given poll captures the true rating equals the confidence level 0.95, the probability that it does not is 0.05.

(b) The probability that all polls capture the true ratings is 0.95^{19} so the probability that at least one misses a true rating is

$$1 - 0.95^{19} \approx 0.62$$

4(a) False, the denominator is not of the form $\sqrt{\chi_r^2/r}$. If we instead had $T = X/|Y|$, this would be t_1.
(b) True by HW5, turn-in problem 1.
(c) False, $\sqrt{X}$ is always nonnegative but the t distribution ranges over all real numbers.
(c) True, the denominator equals $\sqrt{V^2/1}$ where V^2 has a χ_1^2 distribution.
(d) True, $S = ((X+Y)/\sqrt{2})^2$ and $(X+Y)/\sqrt{2} \sim N(0,1)$ so $S \sim \chi_1^2$.

5(a) $L_k = 2z\sigma/\sqrt{n}$ and $L_u = 2ts/\sqrt{n}$.
(b)

$$P(L_u \leq L_k) = P(ts \leq z\sigma) = P\left(\frac{(n-1)s^2}{\sigma^2} \leq \frac{(n-1)z^2}{t^2}\right) = F_{n-1}\left(\frac{(n-1)z^2}{t^2}\right)$$

(c) Intuitively, for large n we will have $t \approx z$ and $s \approx \sigma$ so the intervals will have about the same length. Thus, which one is wider is a coin toss and the probability should approach 1/2.

More formally, since the chi-square distribution is the sum of i.i.d. random variables, it has an approximate normal distribution by the Central Limit Theorem. As the mean in the chi-square distribution equals its degrees of freedom, we have approximately a normal distribution with mean $n-1$ and ask for the probability that it is less than $n-1$ which, by symmetry, equals 1/2. A full proof would require results about uniform convergence in the CLT which we usually don't discuss in undregraduate probability.