Math Stat, solutions to HW3

Practice problems:

1. I. MLE. The likelihood function is

$$L(\mu) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(X_j-\mu)^2/2}$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{j=1}^{n}(X_j-\mu)^2}$$

which has logarithm

$$\log L(\mu) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{j=1}^{n}(X_j-\mu)^2$$

which we differentiate and set equal to 0:

$$\frac{d}{d\mu}\log L(\mu) = \sum_{j=1}^{n}(X_j-\mu) = \sum_{j=1}^{n} X_j - n\mu = 0$$

which gives the MLE $\widehat{\mu} = \bar{X}$.

II. MOME. As $\mu = m_1$, we get the MOME $\widehat{\mu} = \widehat{m}_1 = \bar{X}$.

2. I. MLE. The likelihood function is

$$L(\sigma) = \prod_{j=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-X_j^2/2\sigma^2}$$
$$= \sigma^{-n}\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{j=1}^{n} X_j^2/2\sigma^2}$$

which has logarithm

$$\log L(\sigma) = -n\log\sigma + n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum_{j=1}^{n} X_j^2$$

which we differentiate and set equal to 0:

$$\frac{d}{d\sigma}\log L(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{j=1}^{n} X_j^2$$

$$= \frac{1}{\sigma}\left(-n + \frac{1}{\sigma^2}\sum_{j=1}^{n} X_j^2\right) = 0$$

which gives the MLE

$$\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{j=1}^{n} X_j^2}$$

II. MOME. As $\sigma = \sqrt{m_2 - m_1^2} = \sqrt{m_2}$, we the the MOME

$$\widehat{\sigma} = \sqrt{\widehat{m}_2} = \sqrt{\frac{1}{n}\sum_{j=1}^{n} X_j^2}$$

Turn-in problems

1(a) I. MLE. The likelihood function is

$$L(a) = \prod_{k=1}^{n} aX_k e^{-aX^2/2} = a^n \prod_{k=1}^{n} X_k e^{-a\sum_{k=1}^{n} X_k^2/2}$$

which has logarithm

$$\log L(a) = n\log a + \log(\prod_{k=1}^{n} X_k) - \frac{a}{2}\sum_{k=1}^{n} X_k^2$$

which has derivative

$$\frac{d}{da} = \frac{n}{a} - \frac{1}{2}\sum_{k=1}^{n} X_k^2$$

and setting this equal to 0 gives the MLE

$$\widehat{a} = \left(\frac{1}{2n}\sum_{k=1}^{n} X_k^2\right)^{-1}$$

II. MOME. The first moment is

$$m_1 = E[X] = \int_0^\infty x f(x) dx = \frac{\pi}{\sqrt{2a}}$$

which gives

$$a = \frac{\pi}{2m_1^2}$$

which gives the MOME

$$\widehat{a} = \frac{\pi}{2\widehat{m}_1^2} = \frac{\pi}{2\bar{X}^2}$$

(b) We have $n = 5$, $\bar{X} = 0.64$, and $\sum_{k=1}^n X_k^2 = 2.92$ which gives the MLE

$$\widehat{a} = \left(\frac{1}{10} \cdot 2.92\right)^{-1} = 3.42$$

and the MOME

$$\widehat{a} = \frac{\pi}{2 \cdot 0.64^2} = 3.8$$

Note that although the MLE and the MOME look quite different, their values are in reasonable agreement for our observed sample. In this case, the sample was simulated with a true value of $a = 2$ so both estimators are a bit off, but the sample is small so this is not unusual.

2. The pdf is

$$f_\theta(a) = \begin{cases} 1/2\theta & \text{if } -\theta \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

which gives the likelihood function

$$L(\theta) = \begin{cases} 1/(2\theta)^n & \text{if } -\theta \le \text{ all the } X_k \le \theta \\ 0 & \text{otherwise} \end{cases}$$

Consider the ordered sample $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$. As the likelihood function is positive if and only if

$$-\theta \le \text{ all the } X_k \le \theta$$

which is the same as

$$-\theta \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)} \leq \theta$$

which is the same as

$$\theta \geq -X_{(1)} \quad \text{and} \quad \theta \geq X_{(n)}$$

which is the same as

$$\theta \geq \max(-X_{(1)}, X_{(n)})$$

we get the MLE

$$\widehat{\theta} = \max(-X_{(1)}, X_{(n)})$$

Note that we may have both $X_{(1)}$ and $X_{(n)}$ negative, or both positive, or $X_{(1)}$ negative and $X_{(n)}$ positive.