Math Stat, Solutions to HW6

1. The 4 classes are N, S, W, and E, with the proposed distribution (null hypothesis H_0) $p_1 = ... = p_4 = 1/4$. We have $n = 205+169+177+214 = 765$ which gives $np_k = 765/4 = 191.25$ for all k and we get

Direction	N	S	W	E
Observed	199	169	177	220
Expected	191.25	191.25	191.25	191.25

which gives us the test statistic

$$D = \frac{(199 - 191.25)^2}{191.25} + ... + \frac{(220 - 191.25)^2}{191.25} = 8.3$$

We reject if $D \geq c$ where Table A3 gives $c = 7.815$ ($4 - 1 = 3$ degrees of freedom and $1 - \alpha = 0.95$). As $8.3 > 7.815$, we reject H_0 and conclude that birds do not choose randomly.

2(a) Generally, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ and since we can write T as

$$T = \frac{\bar{X}}{\sigma/\sqrt{n}} - \frac{\mu_0}{\sigma/\sqrt{n}}$$

we can apply the general result that $aY + b \sim N(a\mu + b, a^2\sigma^2)$ if $Y \sim N(\mu, \sigma^2)$ to get the desired distribution for T.

(b) By the result in **(a)** we get

$$1 - \beta(\mu) = P(T > c) = 1 - P(T \leq c) = 1 - \Phi\left(c - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right)$$

(c) With $\alpha = 0.05$ we get $c = 1.64$ (since $\Phi(1.64) = 0.95$). Further, we have $\mu_0 = 10, \mu = 11, \sigma^2 = 1$, and $n = 6$ which gives the probability to reject H_0 as

$$1 - \beta(14) = 1 - \Phi\left(1.64 - \frac{11 - 10}{1/\sqrt{6}}\right) = 1 - \Phi(-0.81) = \Phi(0.81) = 0.79$$

3. We have $n = 1883$, $r = 2$ and $c = 4$. Let p_1 and p_2 be the probabilities of voter and nonvoter, respectively, and let $q_1, ..., q_4$ be the probabilities of no degree,...,college degree. We get the estimates $\widehat{p}_1 = (26 + 223 + 215 + 387)/1883 = 0.45, \widehat{p}_2 = 0.55, \widehat{q}_1 = (26 + 168)/1883 = 0.10, \widehat{q}_2 = 0.35, \widehat{q}_3 = 0.23, \widehat{q}_4 = 0.32$ which gives

$$D = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(X_{ij} - n\widehat{p}_i\widehat{q}_j)^2}{n\widehat{p}_i\widehat{q}_j}$$

$$= \frac{(26 - 1883 \cdot 0.45 \cdot 0.10)^2}{1883 \cdot 0.45 \cdot 0.10} + ... + \frac{(211 - 1883 \cdot 0.35 \cdot 0.32)^2}{1883 \cdot 0.35 \cdot 0.32} = 209$$

The critical value x corresponding to the 5% level is $x = 7.82$ (Table A4, $1 - \alpha = 0.95$, DF $= (r - 1)(c - 1) = 3$). Since our $D > 7.82$ we reject H_0. Voting and education level do not seem to be independent.

4(a) The posterior pdf is

$$f(p|X = n) = \frac{P(X = n|p)f(p)}{c} = \frac{p^n}{c}$$

where

$$c = \int_0^1 P(X = n|p)f(p)dp = \int_0^1 p^n dp = \frac{1}{n+1}$$

which gives the posterior pdf

$$f(p|X = n) = (n + 1)p^n, \quad 0 \leq p \leq 1$$

(b) The Bayes estimator is the mean in the posterior pdf which equals

$$E[p|X = n] = \int_0^1 p(n + 1)p^n dp = (n + 1) \int_0^1 p^{n+1} dp = \frac{n + 1}{n + 2}$$

The relative frequency is $\widehat{p} = 1$.

5. The prior is $B(a, b)$ where we find values of a and b by estimating the mean and variance by the sample mean and sample variance of the previous data. The mean and variance in the $B(a, b)$ distribution are $a/(a + b)$ and $ab/((a + b)^2(a + b + 1))$, respectively, and the data have sample mean 0.0725 and sample variance 0.00049167. Thus, we choose a and b that solve the equations

$$\frac{a}{a+b} = 0.0725$$

and

$$\frac{ab}{(a+b)^2(a+b+1)} = 0.00049167$$

which gives $a = 9.8$ and $b = 125.9$. Since the prior is $B(9.8, 125.9)$ and we have $n = 100$ and observe $k = 4$, the posterior is $B(13.8, 221.9)$ and the Bayes estimator is $13.8/235.7 = 0.059$. The relative frequency is 0.04. Note how the Bayes estimator is between the relative frequency, which is based solely on current data, and the mean in the prior distribution, which is based solely on prior data.