Probability, solutions to HW10

69. We get the correct answer if $Y = n + X$ is rounded to $n$ (and not to $n - 1$ or $n + 1$ or some other integer). This occurs when $X$ is between $-0.5$ and $0.5$ which has probability $P(-0.5 \leq X \leq 0.5) = \Phi(0.5/0.43) - \Phi(-0.5/0.43) = \Phi(1.16) - \Phi(-1.16) = 2\Phi(1.16) - 1 = 0.75.$

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(a) $T \sim N(t + 0.1, 0.01)$.
(b) $T \sim N(11.6, 0.01)$ and $P(T \leq 11.5) = \Phi((11.5 - 11.6)/0.1) = \Phi(-1) = 1 - \Phi(1) = 0.16$.
(c) $P(t - 0.05 \leq T \leq t + 0.05) = \Phi((t + 0.05 - (t + 1))/0.1) - \Phi((t - 0.05) - (t + 0.1))/0.1) = \Phi(-0.5) - \Phi(-1.5) = 0.24$.

71. In each case, start with the cdf and differentiate to get the pdf, recalling that $\Phi'(x) = \varphi(x)$.

(a) Range: $R$. $P(-X \leq x) = P(X \geq -x) = 1 - \Phi(-x) = \Phi(x)$ which has derivative $\varphi(x)$.
(b) Range: $[0, \infty)$. $P(|X| \leq x) = P(-x \leq X \leq x) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1$ which has derivative $2\varphi(x)$.
(c) Range: $[0, \infty)$. $P(X^2 \leq x) = P(-\sqrt{x} \geq X \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = 2\Phi(\sqrt{x}) - 1$ which has derivative $2\varphi(\sqrt{x})/2\sqrt{x} = \varphi(\sqrt{x})/\sqrt{x}$.
(d) Range: $(0, \infty)$. $P(e^X \leq x) = P(X \leq \log x) = \Phi(\log x)$ which has derivative $\varphi(\log x)/x$.

2. This one:

2(a) $P(X \leq 220) = \Phi((220 - 200)/10) = \Phi(2) = 0.98$
(b) $P(X \leq 190) = \Phi((190 - 200)/10) = \Phi(-1) = 1 - \Phi(1) = 0.16$

(c) $P(X > 185) = \Phi(1.5) = 0.93$

(d) $P(X > 205) = 1 - \Phi(0.5) = 0.31$

(e) $P(190 \leq X \leq 210) = \Phi(1) - \Phi(-1) = 0.68$

(f) $P(180 \leq X \leq 210) = \Phi(1) - \Phi(-2) = 0.81$

3. Let $Z = (X - \mu)/\sigma$. Then $Z \sim N(0,1)$ and $P(\mu - 2.58\sigma \leq x \leq \mu + 2.58\sigma) = \Phi(2.58) - \Phi(-2.58) = 2\Phi(2.58) - 1 = 0.99$

4. We have $X \sim N(70,3^2)$ and need to find $a$ and $b$ such that $P(X \leq a) = 0.10$ and $P(X \leq b) = 0.90$. We get $P(X \leq a) = \Phi((a - 70)/3) = 1 - \Phi(-(a - 70)/3) = 0.10$ which gives $\Phi(-(a - 70)/3) = 0.90$ which gives $-(a - 70)/3 = 1.28$ and $a = 70 - 1.28 \cdot 3 = 66$. Further, $P(X \leq b) = \Phi((b - 70)/3) = 0.90$ gives $b = 70 + 1.28 \cdot 3 = 74$.

5. $Z = (\mu + c\sigma - \mu)/\sigma = c$. Thus, the $Z$-score measures by how many standard deviations $X$ differs from $\mu$. If $X$ is smaller than $\mu$, the $Z$-score is negative ($c < 0$).

6. Her $Z$-score on the first test is $(80 - 70)/10 = 1$ and to get the same $Z$-score on the second test, she needs to score 180 points.

7(a) Fix a point $(x, y)$. Then $P(X > x, Y > y)$ is the probability of the region northeast of $(x, y)$ and $F(x, y) = P(X \leq x, Y \leq y)$ is the probability of the region southwest of $(x, y)$. These two probabilities cannot add to 1 because there are also regions northwest and southeast that are not accounted for.

(b) $P(X > x, Y > y) = 1 - \left(F_X(x) + F_Y(y) - F(x, y)\right)$. Note that the area southwest of $(x, y)$ is included in both $F_X(x)$ and $F_Y(y)$ so when these are added, its probability is counted twice which is corrected by subtracting $F(x, y)$.

8(a) Since $(X,Y)$ is uniform on the triangle, $f(x,y)$ must be constant and since the area of the triangle is $1/2$, we get $f(x,y) = 2$.

(b) The range of $X$ is $[0, 1]$. For fixed $x$ in this range we get
\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = 2 \int_0^x \, dy = 2x, \ 0 \leq x \leq 1 \]

The range of \( Y \) is also \([0, 1]\). For fixed \( y \) in this range we get

\[ f_Y(y) = 2 \int_y^1 \, dx = 2(1 - y), \ 0 \leq y \leq 1 \]

Think about why these pdf’s make intuitive sense!