Probability, solutions to HW11

32(a) \( f(x, y) = f_X(x)f_Y(y|x) \) where \( f_X(x) = 1, 0 \leq x \leq 1 \) and \( f_Y(y|x) = \frac{1}{1/x} = x, 0 \leq y \leq 1/x \). Hence, \( f(x, y) = x, 0 \leq x \leq 1, 0 \leq y \leq 1/x \) which is the shaded region below.

(b) For \( 0 \leq y \leq 1 \):
\[
f_Y(y) = \int_0^1 xdx = \frac{1}{2}
\]
and for \( y \geq 1 \):
\[
f_Y(y) = \int_0^{1/y} xdx = \frac{1}{2y^2}
\]

(c) \( P(X > Y) = P((X, Y) \in B) = \int \int_B f(x, y)dxdy \) where \( B \) is the triangular region below

We get
\[
P(X > Y) = \int_{x=0}^1 \int_{y=0}^x xdydx = \int_0^1 x^2dx = 1/3
\]

34(a) Let \( X \) be Billy Bob’s arrival and \( Y \) be Adam’s arrival, in minutes after 12:30. Then \( X \sim \text{unif } [0, 45] \) and \( Y \sim \text{unif } [0, 30] \). Consider the pair \((X, Y)\). By independence, the joint pdf is \( f(x, y) = f_X(x)f_Y(y) = 1/1350, \ 0 \leq x \leq \)
45, 0 ≤ y ≤ 30. We get

\[ P(\text{Billy Bob arrives first}) = P(X < Y) = P((X, Y) \in B) \]

where \( B \) is the region below.

We get

\[
P(X < Y) = \int \int_B f(x, y) \, dx \, dy = \frac{\text{area of } B}{1350} = \frac{450}{1350} = \frac{1}{3}.
\]

(b) The region \( B \) is now the shaded region below and

\[
P((X, Y) \in B) = \frac{800}{1350} = \frac{16}{27} \approx 0.59
\]
36. Check if \( f(x, y) = f_X(x)f_Y(y) \). For example, in (a) we get

\[
f_X(x) = \int_0^1 f(x, y)dy = 4x \int_0^1 ydy = 2x, \ 0 \leq x \leq 1
\]

and \( f_Y(y) = 2y, \ 0 \leq y \leq 1 \) and hence \( f(x, y) = f_X(x)f_Y(y) \) so \( X \) and \( Y \) are independent.

39. The equation has real roots if \( B^2 - 4C \geq 0 \), that is, if \( C \leq B^2/4 \). The probability of the complement is the ratio of two areas: the area bounded by the curve \( y = x^2/4 \) and the line \( y = n \), and the area of the square \([-n, n] \times [-n, n] \). Integration gives the first area as \( 8n\sqrt{n}/3 \) which gives the probability of real roots as

\[
P(\text{real roots}) = 1 - \frac{8n\sqrt{n}/3}{4n^2} = 1 - \frac{2}{3\sqrt{n}}
\]

which goes to 1 as \( n \to \infty \).

47. By independence, the joint pdf is

\[
f(x, y) = f_X(x)f_Y(y) = e^{-x}e^{-y} = e^{-(x+y)}, \ x \geq 0, y \geq 0
\]

and Proposition 3.6.1 gives

\[
E[e^{-(X+Y)/2}] = \int_0^\infty \int_0^\infty e^{-(x+y)/2}e^{-(x+y)}dxdy
\]

\[
= \int_0^\infty \int_0^\infty e^{-3x/2}e^{-3y/2}dxdy = \frac{4}{9}
\]
48. The joint pdf is \( f(x, y) = 1, \ 0 \leq x \leq 1, 0 \leq y \leq 1 \) and Proposition 3.6.1 gives

\[
E[\mathbf{g}(X, Y)] = \int_0^1 \int_0^1 \mathbf{g}(x, y) \, dx \, dy
\]

where \( g(x, y) = xy \) for (a) and \( g(x, y) = x/y \) for (b).

72. In each case, first find \( E[Y \mid X = x] \) and then compute

\[
E[Y] = \int E[Y \mid X = x] f_X(x) \, dx = \int_0^1 E[Y \mid X = x] \, dx
\]

For example, in (d) we have \( E[Y \mid X = x] = x \) \( (Y \mid X = x \sim \exp(\lambda) \) where \( \lambda = 1/x \) which gives

\[
E[Y] = \int_0^1 x \, dx = \frac{1}{2}
\]

2. The cdf is \( F(x) = \int_0^x f(t) \, dt = \frac{3}{8} \int_0^x t^2 \, dt = \frac{x^3}{8} \) which has inverse \( F^{-1}(u) = 2u^{1/3} \). If \( U \sim \text{unif} [0, 1] \), let \( X = 2U^{1/3} \) to get an observation that has the given pdf.

3. Since \((X, Y)\) is uniform on \( B \), the joint pdf is constant \( f(x, y) \equiv c \) where \( 1/c \) is the area of \( B \). The conditional pdf of \( Y \) gives \( X = x \) is

\[
f_Y(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{c}{f_X(x)}
\]

which is constant as a function of \( y \); thus \( Y \mid X = x \) is uniform. Note that \( Y \) itself may or may not be uniform.

4. The joint pdf is \( f(x, y) = 1/\text{area} = 2 \) which gives the marginal of \( X \) as

\[
f_X(x) = \int_0^{1-x} 2 \, dy = 2(1 - x), \ 0 \leq x \leq 1
\]

which gives the coinditional pdf

\[
f_Y(y \mid x) = \frac{1}{1-x}, \ 0 \leq y \leq 1 - x
\]

a uniform distribution on \([0, 1 - x]\).
5. From the variance formula, \( E[X^2] = \text{Var}[X] + (E[X])^2 = \sigma^2 + \mu^2 \).

6(a) By additivity of expected values and Problem 5, we get

\[
E[X_1^2 + \cdots + X_n^2] = E[X_1^2] + \cdots + EX_n^2 = n(\sigma^2 + \mu^2)
\]

(b) By additivity of expected values and variances (for independent random variables), and Problem 3,

\[
E[(X_1 + \cdots + X_n)^2] = \text{Var}[X_1 + \cdots + X_n] + (E[X_1 + \cdots + X_n])^2
\]

\[
= n\sigma^2 + (n\mu)^2 = n\sigma^2 + n^2 \mu^2
\]

7. \( E[X - Y] = \mu - \mu = 0 \) and \( \text{Var}[X - Y] = \text{Var}[X] + \text{Var}[Y] = 2\sigma^2 \). Note that \( \text{Var}[X - Y] \) is not equal to \( \text{Var}[X] - \text{Var}[Y] \).