Probability, solutions to HW7

23. The pmf’s are given in the answer to problem 2.

27. Let \( x \) denote “not 6” and let \( X \) denote your gain. The possible outcome of \( X \) and the corresponding sequences of 3 dice are

- \( X = -1: xxx \), probability \( f(0) = \left(\frac{5}{6}\right)^3 = \frac{125}{216} \)
- \( X = 1, 6xx, x6x, xx6 \), probability \( f(1) = 3 \cdot \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = \frac{75}{216} \)
- \( X = 2, 66x, 6x6, x66 \), probability \( f(2) = 3 \cdot \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right)^2 = \frac{15}{216} \)
- \( X = 3, 666 \), probability \( f(3) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \)

which gives expected gain

\[
E[X] = (-1) \cdot \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = -\frac{17}{216} \approx -0.079
\]

32. Let \( X \) be the breaking point and let \( L \) denote the length of the longest piece. Then \( X \sim \text{unif}[0, 1] \) and \( L = g(X) \) where the function \( g \) is given by

\[
g(x) = \begin{cases} 
1 - x & \text{if } 0 < x \leq 1/2 \\
x & \text{if } x \geq 1/2 
\end{cases}
\]

Since \( f_X(x) = 1 \), we get

\[
E[L] = E[g(X)] = \int_0^1 g(x)dx - \int_{1/2}^1 (1 - x)dx + \int_1^{1/2} xdx = \frac{3}{4}
\]

37. Note that \( E[Y] = \infty \) so we cannot define the variance of \( Y \).

40(b) \( c = 1/(\sqrt{3}(2n + 1)) \to 0 \) as \( n \to \infty \).

2. \( E[X] = (-1) \cdot \frac{35}{38} + 11 \cdot \frac{3}{38} = -2/38, E[X^2] = (-1)^2 \cdot \frac{35}{38} + 11^2 \cdot \frac{3}{38} = 398/38, \text{Var}[X] = 398/38 - (-2/38)^2 = 10.5 \)

3(a) Solve \((-1) \cdot 33/38 + a \cdot 5/38 = -2/38 \) for \( a \) to get the payout \( a = 6.2 \).

(b) \(-3/38 \approx -0.08 \), an expected loss of 8 cents per dollar.
4(a) \( E[X] = \int_0^1 xf(x)dx = \int_0^1 2x^2dx = 2/3 \)

\( \text{Var}[X] = E[X^2] - (E[X])^2 \) where \( E[X] = 2/3 \) and \( E[X^2] = \int_0^1 x^2f(x)dx = \int_0^1 2x^3dx = 1/2; \) thus \( \text{Var}[X] = 1/2 - (2/3)^2 = 1/18.\)

(b) \( E[4\pi X^3/3] = \int_0^1 \frac{4\pi}{3} x^3f(x)dx = \frac{4\pi}{3} \int_0^1 2x^4dx = \frac{8\pi}{15}.\)

5. \( E[V] = 2, \text{Var}[V] = 36/7 \) \( (E[V] = \int_0^2 x^3 f_X(x)dx = (1/2) \int_0^2 x^3dx = 2, E[V^2] = \int_0^2 x^6 f_X(x)dx = 64/7)\)

6(a) \( E[X] = \int_0^1 xf(x)dx = \int_0^1 3x^3dx = 3/4, E[X^2] = \int_0^1 x^2 f(x)dx = \int_0^1 3x^4dx = 3/5, \text{Var}[X] = 3/5 - (3/4)^2 = 3/80.\)

(b) \( E[Y] = \int_0^1 \sqrt{x} f(x)dx = 6/7, E[Y^2] = \int_0^1 x f(x)dx = 3/4, \text{Var}[Y] = 3/4 - (6/7)^2 = 3/196.\)

7(a) \( 1 = c \int_0^\pi \sin x dx = 2c \) which gives \( c = 1/2.\)

(b) \( F(x) = \int_0^x f(t)dt = \frac{1}{2} \int_0^x \sin x dx = \frac{1-\cos x}{2}, \quad 0 \leq x \leq \pi \)

(c) \( E[\csc X] = \frac{1}{2} \int_0^\pi \csc x \sin x dx = \frac{1}{2} \int_0^\pi dx = \frac{\pi}{2} \)