51. The distribution of $X$ is $\text{bin}(n, p)$ where $p$ is the probability that at least 5 flips is required. Hence, $p = P(Y \geq 5)$ where $Y \sim \text{geom}(1/2)$ and since $Y \geq 5$ means that the first 4 flips give tails, we get $P(Y \geq 5) = (1/2)^4 = 1/16$. Hence, $P(X = 0) = (15/16)^n$ and $E[X] = n/16$.

53. To get $X = k$, we need a sequence with $r$ successes and $k - r$ failures, and the sequence must end with a success. Thus, the first $r - 1$ successes can be chosen freely among the first $k - 1$ positions which can be done in $\binom{k-1}{r-1}$ ways. As each such sequence has probability $p^r (1 - p)^{k-r}$, the expressions follows.

54(a) The probability that you miss the bus in a given day is $1/5$ and hence the number of days $X$ until the next missed bus is $\text{geom}(1/5)$. Hence, $P(X = 5) = (4/5)^4 \cdot 1/5 = 0.08$. (b) The number of days $Y$ until you miss for the 5th time has a negative binomial distribution (see problem 53) and hence $P(Y = 10) = \binom{9}{4} (1/5)^5 (4/5)^5 = 0.01$

55(a) $P(X > 0) = 1 - P(X = 0) = 1 - e^{-2} = 0.86$. (b) Note that $Y$ has the same distribution as $X$ conditioned on $X > 0$. Hence the range of $Y$ is 1, 2, ..., and for $k \geq 1$ we get

$$P(Y = k) = P(X = k|X > 0) = \frac{P(X = k) \cap \{X > 0\}}{P(X > 0)} = \frac{e^{-2}2^k}{k!(1 - e^{-2})}$$

where the second to last equality follows since $\{X = k\} \cap \{X > 0\} = \{X = k\}$, in words, if $X$ equals $k \geq 1$, $X$ must be $> 0$. Logically, $X = k \Rightarrow X > 0$ which in terms of events becomes $\{X = k\} \subseteq \{X > 0\}$.

(c) Assuming that consecutive minutes are independent of one another, the number of busy minutes $Y$ until the next idle minute is $\text{geom}(p)$ where $p = P(\text{idle}) = e^{-2}$. The number of busy minutes $B$ before the next idle minutes therefore satisfies $B = Y - 1$ and we get $E[B] = E[Y] - 1 = e^2 - 1.$
56(a) Let $X$ be the number of eggs and condition on $V$: visit and $V^c$. LTP gives

$$P(X = 0) = P(X = 0|V)P(V) + P(X = 0|V^c)P(V^c) = e^{-1}\cdot 0.1 + 1\cdot 0.9 = 0.94$$

2(a) Binomial with $n = 10$ and $p = 0.8$.
(b) Not binomial as trials are not independent (one rainy day makes another more likely)
(c) Not binomial as $p$ changes between months.
(d) Binomial with $n = 10$ and $p = 0.2$.

3. Let $X$ be Ann’s final score, then $X \sim \text{Bin}(10, 1/2)$ and since $1 - p = 1/2$ we have the pmf

$$P(X = k) = \binom{10}{k} \left(\frac{1}{2}\right)^{10}$$

for $k = 0, 1, ..., n$.

(a) The largest probability is when $k = 5$ and equals $P(X = 5) \approx 0.25$.
(b) The event that somebody wins 6–4 is the event that $X = 4$ or $X = 6$ and has probability

$$P(X = 4) + P(X = 6) = \binom{10}{4} \left(\frac{1}{2}\right)^{10} + \binom{10}{6} \left(\frac{1}{2}\right)^{10} \approx 0.41$$

so it is more likely that somebody wins 6–4.

4. First find $n$ and $p$. By the formulas for mean and variance, we have $np = 1$ and $np(1 - p) = 0.9$. We get $1 - p = 0.9$, that is $p = 0.1$, and $n = 1/p = 10$. Finally

$$P(X > 0) = 1 - P(X = 0) = 1 - (1 - p)^n = 1 - 0.9^{10} \approx 0.65$$

5. Let $X$ be the number of drug interactions. Then $X \sim \text{bin}(n, p)$ where $p = 0.01$ and $n$ is the number of pairs of medications, that is, $n = \binom{7}{2} = 21$. This gives the probability of some interaction as
\[ P(X > 0) = 1 - P(X = 0) = 1 - (1 - p)^n = 1 - 0.99^{21} = 0.19 \]

6(a) Yes, \( X = 0 \) means that there are no successes in \( n \) trials, \( Y > n \) means that the first success occurs after the \( n \)th trial; same thing.

(b) No, \( X = 1 \) means that there is exactly one success in \( n \) trials, \( Y \leq n \) means that the first success occurs in the first \( n \) trials but there could still be more than one success.

(c) \( P(Y \leq n) = P(X \geq 1) \)

8(a) Assume that hurricanes hit according to a Poisson process with rate \( \lambda \) (hits per month). The number of hits in a season is then \( X \sim \text{Poi}(6\lambda) \). Since the probability of 0 hits is 1/2 we get

\[ \frac{1}{2} = P(X = 0) = e^{-6\lambda} \]

which gives \( \lambda = -\frac{\log 1/2}{12} \approx 0.115 \), the mean number of hits in a given month.

(b) The mean number of hits in 2 months is \( 2 \cdot 0.115 = 0.23 \); hence \( X \sim \text{Poi}(0.23) \).

(c) Since \( Y \) counts the number of “successes” (hit-free months), it has a binomial distribution with \( n = 6 \). If \( X \) denotes the number of hurricanes in a given month, \( X \sim \text{Poi}(0.115) \) and the success probability is \( p = P(X = 0) = e^{-0.115} = 0.89 \). Thus, \( Y \sim \text{bin}(6, 0.89) \).

9(a) Assume that meteorites hit according to a Poisson process with rate \( \lambda \) (hits per millennium). Let \( X \) be the number of hits within the next 1,000 years. Then \( X \sim \text{Poi}(1) \) and we get \( P(X > 0) = 1 - P(X = 0) = 1 - e^{-1} = 0.63 \).

(b) \( P(X \geq 2) = 1 - P(X \leq 1) = 1 - (P(X = 0) + P(X = 1)) = 1 - (e^{-1} + e^{-1}) = 0.26 \)

(c) Here \( X \sim \text{Poi}(0.1) \) and \( P(X = 0) = e^{-0.1} = 0.9 \).