Probability, solutions to HW10

Practice problems

2. This one:

6. Recall that $F_X(x) = F(x, \infty)$ and $F_Y(y) = F(\infty, y)$. If the range is unbounded, we interpret these as limits as $x$ and $y$ go to infinity; if the range is bounded we insert the upper bound of the range. For example:

(a) $F_X(x) = F(x, \infty) = \lim_{y \to \infty} (1 - e^{-x} - e^{-y} + e^{-(x+y)}) = 1 - e^{-x}$. Note that this is an exponential distribution with $\lambda = 1$ and that $Y$ has the same distribution and that $X$ and $Y$ are independent since the joint cdf equals the product of the marginal cdfs.

(c) $F_X(x) = F(x, \infty) = F(x, 1) = 2x, 0 \leq x \leq 1/2$ and $F_Y(y) = F(\infty, y) = F(1/2, y) = y, 0 \leq y \leq 1$.

34(a) Let $X$ be Billy Bob's arrival and $Y$ be Adam's arrival, in minutes after 12:30. Then $X \sim \text{unif } [0, 45]$ and $Y \sim \text{unif } [0, 30]$. Consider the pair $(X, Y)$. By independence, the joint pdf is $f(x, y) = f_X(x)f_Y(y) = 1/1350, \ 0 \leq x \leq 45, 0 \leq y \leq 30$. We get

$$P(\text{Billy Bob arrives first}) = P(X < Y) = P((X, Y) \in B)$$

where $B$ is the region below.

We get
\[ P(X < Y) = \int\int_B f(x, y) \, dx \, dy = \frac{\text{area of } B}{1350} = \frac{450}{1350} = \frac{1}{3}. \]

(b) The region \( B \) is now the shaded region below and

\[ P((X, Y) \in B) = \frac{800}{1350} = \frac{16}{27} \approx 0.59 \]

36. Check if \( f(x, y) = f_X(x)f_Y(y) \). For example, in (a) we get

\[ f_X(x) = \int_0^1 f(x, y) \, dy = 4x \int_0^1 y \, dy = 2x, \ 0 \leq x \leq 1 \]

and \( f_Y(y) = 2y, \ 0 \leq y \leq 1 \) and hence \( f(x, y) = f_X(x)f_Y(y) \) so \( X \) and \( Y \) are independent.

**Turn-in problems**
1. Remember that $F(x, y)$ is the integral of the joint pdf over the region $(-\infty, x] \times (-\infty, y]$, then add and subtract to get the answer.

3(a) $F(x, x) = P(X \leq x, Y \leq x) \to 1$, (b) $F(-x, x) = P(X \leq -x, Y \leq x) \leq P(X \leq -x) \to 0$, (c) 0.

16(a)

\[
1 = \int \int f(x, y)dxdy = c \int_{x=0}^{1} \int_{y=0}^{x} (x - y)dydx
\]
\[
= c \int_{x=0}^{1} \left[ xy - \frac{y^2}{2} \right]_{y=0}^{x} dx
\]
\[
= c \int_{0}^{1} \frac{x^2}{2} dx = \frac{c}{6}
\]

which gives $c = 6$.

(b)

\[
P(X > \frac{1}{2}, Y \leq \frac{1}{2}) = \int_{x=1/2}^{1} \int_{y=0}^{\sqrt{\frac{1}{2}}} 6(x - y)dxdy = \frac{3}{4}
\]

(c)

\[
P(X \leq 2Y) = \int_{x=1/2}^{1} \int_{y=x/2}^{x} 6(x - y)dxdy = \frac{1}{4}
\]

(d)

\[
f_X(x) = \int f(x, y)dy = \int_{0}^{x} 6(x - y)dy = 6 \left[ xy - \frac{y^2}{2} \right]_{y=0}^{x} = 3x^2, \ 0 \leq x \leq 1
\]

\[
f_Y(y) = \int f(x, y)dx = \int_{y}^{1} 6(x - y)dx = 6 \left[ \frac{x^2}{2} - xy \right]_{y}^{1} = 3(y - 1)^2, \ 0 \leq y \leq 1
\]

2(a) Since $(X, Y)$ is uniform on the triangle, $f(x, y)$ must be constant and since the area of the triangle is $1/2$, we get $f(x, y) = 2$. 


(b) The range of $X$ is $[0, 1]$. For fixed $x$ in this range we get

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = 2 \int_0^x dy = 2x, \ 0 \leq x \leq 1$$

The range of $Y$ is also $[0, 1]$. For fixed $y$ in this range we get

$$f_Y(y) = 2 \int_y^1 dx = 2(1 - y), \ 0 \leq y \leq 1$$

Think about why these pdf’s make intuitive sense.