Probability, Solutions to HW3

Practice problems:

31(a) There are $\binom{10}{3}$ ways to choose the 3 numbers. To get the smallest number equal to 4, the number 4 must be chosen (and there is trivially $\binom{1}{1} = 1$ such choice) and the remaining 2 numbers must be chosen among 5 - 10 for which there are $\binom{6}{2}$ ways.

35. This is the matching problem so the probability is approximately $1 - e^{-1} \approx 0.63$.

37. False. Take any independent events $A$ and $B$ with $P(A) \neq 1/2$. Then $P(A|B) + P(A|B^c) = 2P(A) \neq 1$.

41. Let $A$ and $B$ be the events that he wins the first and second elections, respectively. Then

(a) $P(A \cap B) = P(A)P(B|A) = 0.60 \cdot 0.75 = 0.45$

(b) $P(A \cap B^c) = P(A)P(B^c|A) = 0.60 \cdot 0.25 = 0.15$

(c) $P(A|B) = P(A \cap B)/P(B) = 0.45/0.50 = 0.90$

(d)

$$P(B|A^c) = \frac{P(A^c \cap B)}{P(A^c)} = \frac{P(A^c|B)P(B)}{P(A^c)}$$

$$= \frac{(1 - P(A|B))P(B)}{1 - P(A)} = \frac{0.10 \cdot 0.5}{0.4} = 0.125$$

Turn-in problems:

1. Let $F$: flood-damaged and $E$: engine problems. Then $P(F) = 0.05$, $P(F^c) = 0.95$, $P(E|F) = 0.80$, and $P(E|F^c) = 0.10$ and LTP gives

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = 0.80 \cdot 0.05 + 0.10 \cdot 0.95 = 0.135$$
36(a) Let $P(j)$ be the probability of $j$ matches. From class we know that

$$P(0) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and as we also have $P(0) = \frac{n_0}{n!}$, we get

$$n_0 = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(b) The probability of exactly $j$ matches is $P(j) = \frac{n_j}{n!}$ where $n_j$ is the number of sequences with exactly $j$ matches. Now fix a particular set of $j$ numbers, such as $\{1, 2, ..., j\}$ and note that in order to match exactly those numbers, there must be 0 matches among the other $n - j$ numbers. Thus, there are $(n-j)_0$ sequences that match the numbers $\{1, 2, ..., j\}$. Fix another set of $j$ numbers and again there are $(n-j)_0$ sequences that match those numbers. As there are $\binom{n}{j}$ ways to choose $j$ out of $n$ numbers, there are a total of $\binom{n}{j}(n-j)_0$ sequences that have exactly $j$ matches. Hence

$$n_j = \binom{n}{j}(n-j)_0 = \frac{n!}{j!(n-j)!} \frac{n-j}{(n-j)!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}$$

and we get

$$P(j) = \frac{n_j}{n!} = \frac{1}{j!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}$$

As $n \to \infty$ we have $(n-j) \to \infty$ as well and

$$\sum_{k=1}^{n-j} \frac{(-1)^k}{k!} \to e^{-1}$$

and we get

$$P(j) \to \frac{e^{-1}}{j!}$$

as $n \to \infty$. Thus, for large $n$ and moderate $j$, we get

$$P(j) \approx \frac{e^{-1}}{j!}$$
The “moderate \( j \)” qualification is necessary because we need \( n - j \) to be large. For example, obviously \( P(n-1) = 0 \) (if \( n-1 \) numbers match, the \( n \)th must also match).

**59(a)** \( P(A) = 1/2, P(B) = 2/3, P(C) = 3/36, P(A \cap B \cap C) = 1/36 \).

**b)** No, because for example \( A \) and \( C \) are not independent: \( P(A) = 1/2 \) but \( P(A|C) = 1 \). Note that \( P(A \cap B \cap C) = P(A)P(B)P(C) \) but this is not enough for independence of all three events because we do not have pairwise independence.

**72.** Let \( S \): survive delivery and \( C \): cesarean section. By LTP:

\[
0.993 = P(S) = P(S|C)P(C) + P(S|C^c)P(C^c) = 0.987 \cdot 0.15 + P(S|C^c) \cdot 0.85
\]

which gives \( P(S|C^c) = 0.994 \).