Probability, solutions to HW9

77(a) \( P(X \leq 220) = \Phi((220 - 200)/10) = \Phi(2) = 0.98 \)
(b) \( P(X \leq 190) = \Phi((190 - 200)/10) = \Phi(-1) = 1 - \Phi(1) = 0.16 \)
(c) \( P(X > 185) = \Phi(1.5) = 0.93 \)
(d) \( P(X > 205) = 1 - \Phi(0.5) = 0.31 \)
(e) \( P(190 \leq X \leq 210) = \Phi(1) - \Phi(-1) = 0.68 \)
(f) \( P(180 \leq X \leq 210) = \Phi(1) - \Phi(-2) = 0.81 \)

78. Done in class.

79. We have \( X \sim N(70, 3^2) \) and need to find \( a \) and \( b \) such that \( P(X \leq a) = 0.10 \) and \( P(X \leq b) = 0.90 \). We get \( P(X \leq a) = \Phi((-a - 70)/3) = 1 - \Phi((-a - 70)/3) = 0.10 \) which gives \( -(-a - 70)/3 = 1.28 \) and \( a = 70 - 1.28 \cdot 3 = 66 \). Further, \( P(X \leq b) = \Phi((b - 70)/3) = 0.90 \) gives \( b = 70 + 1.28 \cdot 3 = 74 \).

80. \( Z = (\mu + c\sigma - \mu)/\sigma = c \). Thus, the \( Z \)-score measures by how many standard deviations \( X \) differs from \( \mu \). If \( X \) is smaller than \( \mu \), the \( Z \)-score is negative \( (c < 0) \).

84(a) \( T \sim N(t + 0.1, 0.01) \).
(b) \( T \sim N(11.6, 0.01) \) and \( P(T \leq 11.5) = \Phi((11.5 - 11.6)/0.1) = \Phi(-1) = 1 - \Phi(1) = 0.16 \).
(c) \( P(t - 0.05 \leq T \leq t + 0.05) = \Phi((t + 0.05 - (t + 0.1))/0.1) - \Phi((t - 0.05) - (t + 0.1))/0.1) = \Phi(-0.5) - \Phi(-1.5) = 0.24 \).

85. In each case, start with the cdf and differentiate to get the pdf, recalling that \( \Phi'(x) = \varphi(x) \).

(a) Range: \( R \). \( P(-X \leq x) = P(X \geq -x) = 1 - \Phi(-x) = \Phi(x) \) which has derivative \( \varphi(x) \).
(b) Range: \( [0, \infty) \). \( P(|X| \leq x) = P(-x \leq X \leq x) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1 \) which has derivative \( 2\varphi(x) \).
(d) Range: \( (0, \infty) \). \( P(e^X \leq x) = P(X \leq \log x) = \Phi(\log x) \) which has derivative \( \varphi(\log x)/x \).

Turn-in problems
81. The $z$-scores are

$$Z_A = \frac{24 - 20}{2} = 2$$

and

$$Z_B = \frac{48 - 40}{8} = 1$$

so the 48-pound $A$-fish is more extreme.

82. Her $Z$-score on the first test is $(80 - 70)/10 = 1$ and to get the same $Z$-score on the second test, she needs to score 180 points.

83. We get the correct answer if $Y = n + X$ is rounded to $n$ (and not to $n - 1$ or $n + 1$ or some other integer). This occurs when $X$ is between $-0.5$ and $0.5$ which has probability $P(-0.5 \leq X \leq 0.5) = \Phi(0.5/0.43) - \Phi(-0.5/0.43) = \Phi(1.16) - \Phi(-1.16) = 2\Phi(1.16) - 1 = 0.75$.

85(c) Range: $[0, \infty)$. $P(X^2 \leq x) = P(-\sqrt{x} \geq X \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = 2\Phi(\sqrt{x}) - 1$ which has derivative $2\varphi(\sqrt{x})/2\sqrt{x} = \varphi(\sqrt{x})/\sqrt{x}$.