Probability, Solutions to HW1

Practice problems:

19(a) There are 26^5 ways to choose 5 lowercase letters and 10^2 ways to choose 2 digits. For each such choice there are $\binom{7}{2}$ different passwords (choose position for the 2 digits) so the total number of possible passwords is $26^5 10^2 \binom{7}{2} \approx 25$ billion.

(b) There are $(26)_5$ ways to choose 5 lowercase letters and $(10)_2$ ways to choose 2 digits. For each such choice there are $\binom{7}{2}$ different passwords (choose position for the 2 digits) so the total number of possible passwords is $(26)_5(10)_2\binom{7}{2} \approx 10$ billion.

20. The total number of plates is $23^3 10^3$ which is the denominator in each case. The number of outcomes that satisfy the event are: **(a)** $23 \cdot 22 \cdot 21 \cdot 10^3$, **(b)** $23^3 \cdot 10 \cdot 9 \cdot 8$, **(c)** $23 \cdot 1 \cdot 1 \cdot 10^3$, **(d)** $26^3 \cdot 5^3$, $23 \cdot 22 \cdot 21 \cdot 10 \cdot 1 \cdot 1$.

23(a) Choose positions for the two As, this can be done in $\binom{4}{2} = 6$ ways; the Bs are then placed in the two remaining places (formally in $\binom{2}{2}$ ways). Thus, there are 6 possible band names.

(b) First choose positions for the As: $\binom{n}{n_A}$ ways. Next, choose positions for the Bs: $\binom{n-n_A}{n_B}$ ways, and so on. The final expression becomes

$$\binom{n}{n_A}\binom{n-n_A}{n_B}\cdots\binom{n-n_A-\cdots-n_Y}{n_Z}$$

Now use the definition of $\binom{n}{k}$ and notice how all factors of the type "n minus something" in the denominator also appear in the numerator to arrive at the expression in the problem.

31(a) There are $\binom{10}{3}$ ways to choose the 3 numbers. To get the smallest number equal to 4, the number 4 must be chosen (and there is trivially $\binom{1}{1} = 1$ such choice) and the remaining 2 numbers must be chosen among $5-10$ for which there are $\binom{6}{2}$ ways.

2. With $n = 2, k = 2$, we get $P(A) = 1/2$ (correctly) in the first case and $P(A) = 2/3$ (incorrectly) in the second case. The ordered outcomes $(1,1), (1,2), (2,1), (2,2)$ are equally likely whereas the unordered outcomes $\{1,1\}, \{1,2\}, \{2,2\}$ are not (the middle one is twice as likely as each of the others).

In general, the first way, the one that takes order into account, is correct because each ordered outcome has the same probability. The unordered outcomes in the second approach do not have the same probabilities. For example, in the birthday problem, if we label the days $1, 2, ..., 365$, the unordered outcome $\{1, 1, ..., 1\}$ corresponds to the ordered outcome $(1, 1, ..., 1)$; thus these two have the same probability. However, the unordered outcome $\{2, 1, 1, ..., 1\}$ corresponds to the 365 different ordered outcomes $(2, 1, 1, ..., 1)$, $(1, 2, 1, ..., 1), ..., (1, 1, ..., 1, 2)$ and is thus 365 times as likely as $\{1, 1, ..., 1\}$.

Turn-in problems:

1. The 5th card can be chosen in 52 ways. The ranks for the 2 pairs must avoid the rank of the 5th card and can thus be chosen in $\binom{12}{2}$ ways. The suits can then be chosen in $\binom{4}{2}\binom{4}{2}$ ways, giving a total number of 2-pair hands of $52\binom{12}{2}\binom{4}{2}\binom{4}{2} = 123,552$. Previously we got $\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 44 = 123,552$, same result.

2. There is a total of $\binom{52}{13}$ ways to choose a bridge hand. The suit distribution 13–0–0–0 can be chosen in 4 ways (each of the 4 suits) and thus

$$P(13 - 0 - 0 - 0) = \frac{4}{\binom{52}{13}} \approx 6 \cdot 10^{-12}$$

For 5–4–3–1, we need to choose 5 out of 13 in one suit, 4 out of 13 in another, 3 out of 13 in a third, and 1 out of 13 in the fourth. The first suit can be chosen in $\binom{4}{1} = 4$ ways, the second in $\binom{3}{1} = 3$ ways and so on which gives $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways (you can also note that the 4 suits can be ordered in $4! = 24$ different ways). We get

$$P(5 - 4 - 3 - 1) = \frac{24 \cdot \binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} \approx 0.13$$

Note that 5–4–3–1 is more likely than the "expected" suit distribution 4–3–3–3.

3. The Venn diagram in the problem does not depict the most general case. For example, there is no intersection between A and D that does not intersect either B or C. Thus, the Venn diagram depicts a special case in which the intersections $A \cap D \cap C^c \cap B^c$ and $B \cap C \cap A^c \cap D^c$ are empty. No general formulas can be inferred from this Venn diagram. Note that the most general Venn diagram of n events has 2^n disjoint pieces (counting the one outside the events). Think about why this is the case!