# MATH 3334, Probability, Solutions to Test 1

**1.** (a) and (c) are always true.

For (a), note that $A = (A \cap B^c) \cup (A \cap B)$ and apply the third axiom
(b) is only true for independent events
For (c), note that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ where $P(A \cap B) \leq 1$
(d) for a counterexample, let $A$ and $B$ be independent with $P(A) \neq 1/2$
(e), for a counterexample, let $P(A) < 1$ and $B = A$

**2(a)** $P(A \cup B) = P(A) + P(B) - P(A)P(B) = 1/2$, $P(A \cap B) = P(A)P(B) = 1/12$
**(b)** $P(A \cup B) = P(A) + P(B) = 7/12$, $P(A \cap B) = 0$
**(c)** Draw a Venn diagram to argue that $A^c \cap B = \emptyset$ implies that $B \subseteq A$. Thus, $P(A \cup B) = P(A) = 1/3$ and $P(A \cap B) = P(B) = 1/4$.

**3.** To avoid aces, you have 12 cards in each suit to choose from. The hearts can be chosen in $\binom{12}{5}$ ways, the diamonds in $\binom{12}{3}$ ways and the remaining 5 cards in $\binom{24}{5}$ ways. As the total number of bridge hands is $\binom{52}{13}$, the answer is

$$\frac{\binom{12}{5}\binom{12}{3}\binom{24}{5}}{\binom{52}{13}} \approx 0.01$$

**4.** Let $D$ denote having the disorder, let $A$ and $B$ be the events to have gene $A$ and $B$, respectively.

**(a)** By independence $P(A \cap B^c) = P(A)P(B^c) = 0.01 \cdot 0.99 = 0.099$.

**(b)** By Bayes rule

$$P(A|+) = \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|A^c)P(A^c)} = \frac{1 \cdot 0.01}{1 \cdot 0.01 + 0.01 \cdot 0.99} = \frac{100}{199} \approx 0.5$$

**(c)** As in **(b)**, but with $P(B) = 0.01 \cdot 100/199 = 1/199$ we get

$$P(B|+) = \frac{P(+|B)P(B)}{P(+|B)P(B) + P(+|B^c)P(B^c)} = \frac{1 \cdot 1/199}{1 \cdot 1/199 + 0.01 \cdot 198/199}$$

$$= (\frac{100}{199})^2 \approx 0.25$$

which can also be realized because you have a 100/199 probability of testing positive twice in independent tests.

**5.** Let $C$ be the event to get correct text and condition on the events $A_k$: exactly $k$ As are chosen, $k = 1, 2, 3, 4$. LTP gives

$$P(C) = \sum_{k=1}^{4} P(C|A_k)P(A_k)$$

where $P(C|A_1) = 1/24, P(C|A_2) = 1/12, P(C|A_3) = 1/4, P(C|A_4) = 1$, or more compactly

$$P(C|A_k) = \frac{k!}{4!}$$

The 4 cases have probabilities

$$P(A_k) = \frac{\binom{4}{k}\binom{3}{4-k}}{\binom{7}{4}}$$

which gives $P(C) = 17/105$.

**6.** Condition on the next two changes. With probability $2/3 \cdot 2/3 = 4/9$, both changes are both completions of jobs and the busy period is over without any lost jobs. With probability $1/3 \cdot 1/3 = 1/9$, both changes are arrivals of new jobs and the second of these jobs is lost. Finally, with probability $1 - (4/9 + 1/9) = 4/9$, there is one completion and one arrival, in any order, and the system starts over at two jobs. Formally, let $A$ be the event that no jobs are lost before the current busy period ends and let

$B_1 = \{\text{next two changes are completions}\}$
$B_2 = \{\text{next two changes are arrivals}\}$
$B_3 = \{\text{next two changes are one of each}\}$

Now let $q = P(A)$ and note that $P(A|B_1) = 1$ and $P(A|B_2) = 0$. In the third case, the system starts over with 2 jobs and thus $P(A|B_3) = q$. LTP gives

$$q = P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= 1 \cdot 4/9 + 0 \cdot 1/9 + q \cdot 4/9$$

and we get the equation $q = 4/9 + 4q/9$ which has solution $q = 4/5$. Note that this problem is essentially the same as the tennis problem we did in class.