Probability, Solutions to HW2

Practice problems:

31(a) There are $\binom{10}{3}$ ways to choose the 3 numbers. To get the smallest number equal to 4, the number 4 must be chosen (and there is trivially $\binom{1}{1} = 1$ such choice) and the remaining 2 numbers must be chosen among $5 - 10$ for which there are $\binom{6}{2}$ ways.

35. This is the matching problem so the probability is approximately $1 - e^{-1} \approx 0.63$.

36(a) Let $P(j)$ be the probability of j matches. From class we know that

$$P(0) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and as we also have $P(0) = n_0/n!$, we get

$$n_0 = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(b) The probability of exactly j matches is $P(j) = n_j/n!$ where n_j is the number of sequences with exactly j matches. Now fix a particular set of j numbers, such as $\{1, 2, ..., j\}$ and note that in order to match exactly those numbers, there must be 0 matches among the other $n - j$ numbers. Thus, there are $(n-j)_0$ sequences that match the numbers $\{1, 2, ..., j\}$. Fix another set of j numbers and again there are $(n - j)_0$ sequences that match those numbers. As there are $\binom{n}{j}$ ways to choose j out of n numbers, there are a total of $\binom{n}{j}(n - j)_0$ sequences that have exactly j matches. Hence

$$n_j = \binom{n}{j}(n - j)_0 = \frac{n!}{j!(n - j)!}(n - j)! \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}$$

and we get

$$P(j) = \frac{n_j}{n!} = \frac{1}{j!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}$$

As $n \to \infty$ we have $(n-j) \to \infty$ as well and

$$\sum_{k=1}^{n-j} \frac{(-1)^k}{k!} \to e^{-1}$$

and we get

$$P(j) \to \frac{e^{-1}}{j!}$$

as $n \to \infty$. Thus, for large n and moderate j, we get

$$P(j) \approx \frac{e^{-1}}{j!}$$

The "moderate j" qualification is necessary because we need $n-j$ to be large. For example, obviously $P(n-1) = 0$ (if $n-1$ numbers match, the nth must also match).

37. False. Take any independent events A and B with $P(A) \neq 1/2$. Then $P(A|B) + P(A|B^c) = 2P(A) \neq 1$.

41. Let A and B be the events that he wins the first and second elections, respectively. Then

(a) $P(A \cap B) = P(A)P(B|A) = 0.60 \cdot 0.75 = 0.45$

(b) $P(A \cap B^c) = P(A)P(B^c|A) = 0.60 \cdot 0.25 = 0.15$

(c) $P(A|B) = P(A \cap B)/P(B) = 0.45/0.50 = 0.90$

(d)

$$P(B|A^c) = \frac{P(A^c \cap B)}{P(A^c)} = \frac{P(A^c|B)P(B)}{P(A^c)}$$

$$= \frac{(1 - P(A|B))P(B)}{1 - P(A)} = \frac{0.10 \cdot 0.5}{0.4} = 0.125$$

72. Let S: survive delivery and C: cesarean section. By LTP:

$$0.993 = P(S) = P(S|C)P(C) + P(S|C^c)P(C^c) = 0.987 \cdot 0.15 + P(S|C^c) \cdot 0.85$$

which gives $P(S|C^c) = 0.994$.

73. Let A be the event "no heads" and let B_k be the event that the number on the die is k, $k = 1, 2, ..., 6$. LTP gives

$$P(A) = \sum_{k=1}^{6} P(A|B_k)P(B_k)$$

where $P(B_k) = 1/6$ for all k, and

$$P(A|B_k) = (1/2)^k$$

which gives

$$P(A) = \frac{1}{6}\sum_{k=1}^{6}(1/2)^k$$

$$= \frac{1}{6}\left(\frac{1-(1/2)^7}{1-1/2} - 1\right) \approx 0.164$$

Recall the identity

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

Turn-in problems:

1. Let F: flood-damaged and E: engine problems. Then $P(F) = 0.05$, $P(F^c) = 0.95$, $P(E|F) = 0.80$, and $P(E|F^c) = 0.10$ and LTP gives

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = 0.80 \cdot 0.05 + 0.10 \cdot 0.95 = 0.135$$

59(a) The sample space is the set of the 36 pairs (i, j) where i and j range between 1 and 6: $S = \{(i, j) : 1 \le i \le 6, 1 \le j \le 6\}$. We get $P(A) = 1/2$ and $P(B) = 2/3$. For C, note that there are three outcomes that give the sum 10: $(4, 6), (5, 5)$, and $(6, 4)$, and hence $P(C) = 3/36$. The intersection

$A \cap B \cap C$ contains the single outcome $(5, 5)$ so we have $P(A \cap B \cap C) = 1/36$. **(b)** No, because for example A and C are not independent: $P(A) = 1/2$ but $P(A|C) = 1$. Note that $P(A \cap B \cap C) = P(A)P(B)P(C)$ but this is not enough for independence of all three events because we do not have pairwise independence.

76. Let C: correct text and use LTP with the three cases I : three As, II: (exactly) two letters are the same, and III: all letters are different. Then $P(C|I) = 1, P(C|II) = 1/3$, and $P(C|III) = 1/3! = 1/6$. Moreover, there are $\binom{8}{3} = 56$ ways to choose three letters and we get $P(I) = 1/56$. For case II, note that the two letters can be two As or two Ss. There are $\binom{3}{2} \cdot \binom{5}{1} = 15$ ways to choose two As and $\binom{2}{2}\binom{6}{1} = 6$ ways to choose two Ss and hence $P(II) = 21/56$, which also gives $P(III) = 34/56$. Taken together we get

$$P(C) = 1 \cdot \frac{1}{56} + \frac{1}{3} \cdot \frac{21}{56} + \frac{1}{6} \cdot \frac{34}{56} \approx 0.24$$