Probability, solutions to Test 2

1(a) True. $p(x) = P(X = x) \leq 1$ by the axioms of probability.
(b) False. Counterexample: $X \sim$ unif $[0, 1/2]$ has $f(x) = 2, 0 \leq x \leq 1/2$.
(c) True. $F(x) = P(X \leq x) \leq 1$ by the axioms of probability.
(d) True. $F(x) = P(X \leq x) \leq 1$ by the axioms of probability.
(e) True, $n - X$ counts the number of failures.
(f) False. The range of $2X$ is $0, 2, ..., 2n$ whereas a bin$(2n, p)$ distribution has range $0, 1, 2, ..., 2n$.
(g) False. Any nonnegative function that integrates to 1 over the unit square is a possible joint pdf and it does not have to factor into the marginals. For one example, take $f(x, y) = x + y$ which has marginals $f(x) = x + 1/2$ and $f(y) = y + 1/2$ so X and Y are not independent.
(h) True. X and Y restrict each other's ranges.

3(a) The variance is $(1-p)/p^2 = 6$ which gives $p = 1/3$ and $P(X = 0) = 0$ and $P(0 < x \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{5}{9}$

(b) The variance is $1/\lambda^2$ which gives $\lambda = 2$ and $P(X = 0) = 0$ (continuous random variable) and $P(0 < X \leq 2) = F(2) = 1 - e^{-2 \cdot 2} = 0.98$.

(c) $P(X = 0) = \exp(-2) = 0.14$ and $P(0 < x \leq 2) = P(X = 1) + P(X = 2) = 4\exp(-2) = 0.54$

(d) $P(X = 0) = 0$ and $P(0 < x \leq 2) = \Phi(1/\sqrt{2}) - \Phi(-1/\sqrt{2}) = 0.52$

3. As

$$1 = \int_0^2 f(x)dx = c\int_0^2 x^2 dx = 8c/3$$

we get $c = 3/8$. Next, as $V = 4\pi R^3/3$, we get

$$E[V] = \int_0^2 \frac{4\pi x^3}{3}\frac{3}{8}x^2 dx = \frac{\pi}{2}\int_0^2 x^5 dx = \frac{16\pi}{3} \approx 16.8$$

and

$$E[V^2] = \int_0^2 \left(\frac{4\pi x^3}{3}\right)^2 \frac{3}{8}x^2 dx = \frac{1024\pi^2}{27}$$

which gives

$$\mathrm{Var}[V] = \frac{1024\pi^2}{27} - \left(\frac{16\pi}{3}\right)^2 = \frac{256\pi^2}{27} \approx 94$$

4(a) Assume that hurricanes hit according to a Poisson process with rate λ (hits per month). The number of hits in a season is then $X \sim \mathrm{Poi}(6\lambda)$. Since the probability of 0 hits is 1/2 we get

$$\frac{1}{2} = P(X = 0) = e^{-6\lambda}$$

which gives $\lambda = \dfrac{\log 2}{6} \approx 0.116$, the mean number of hits in a given month.

(b) The mean number of hits in 2 months is $2 \cdot 0.116 = 0.23$; hence $X \sim \mathrm{Poi}(0.23)$

(c) Let T be the time until the next hit. Then $T \sim \exp(\lambda)$ and by the memoryless property, it does not matter when the previous hit occurred so we get $P(T \leq 1) = 1 - e^{-0.116} = 0.11$.

5(a) As $\displaystyle\int_0^1 \int_{x^2}^x dydx = \frac{1}{6}$, we get $f(x, y) = 6$.

(b) The region to integrate f over is $\{(x, y) : 1/2 \leq x \leq 1, x^2 \leq y \leq x\}$ and we get

$$P(X > 1/2) = 6 \int_{x=1/2}^1 \int_{y=x^2} xdydx = \ldots = \frac{1}{2}$$

(c) The marginal pdf of X is

$$f_X(x) = \int f(x, y)dy = \int_{x^2}^x 6ydy = 6(x - x^2),\ 0 \leq x \leq 1$$

and that of Y is

$$f_Y(y) = \int f(x, y)dx = \int_y^{\sqrt{y}} 6dx = 6(\sqrt{y} - y),\ 0 \leq y \leq 1$$

(d) The conditional pdf of X given $Y = y$ is

$$f_X(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{\sqrt{y} - y},\ y \leq x \leq \sqrt{y}$$

and that of Y given $X = x$ is

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{x - x^2}, \; x^2 \le y \le x$$

(e) Both are uniform because there is no dependence of $f_X(x|y)$ on the argument x, and no dependence of $f_Y(y|x)$ on the argument y.

6(a) By the formula for the expected value of a function of X:

$$E[F(X)] = \int_{-\infty}^{\infty} F(x) f(x) dx$$

Now make the change of variables $y = F(x)$ which gives $dy = f(x)dx$ and $(-\infty, \infty) \to (0,1)$ to get

$$E[F(X)] = \int_0^1 y dy = \frac{1}{2}$$

(b) Use the continuous version of LTP to get

$$P(Y \le X) = \int P(Y \le x) f_X(x) dx = \int F(x) f(x) dx = E[F(X)]$$

which equals 1/2, by **(a)**.

7(b) Start with the joint cdf G of (m, M). For $x \le y$, we get $G(x,y) = P(m \le x, M \le y)$. Letting F be the joint cdf of (X,Y) and recalling that $F(x,y)$ gives the probability of the rectangle "southwest" of the point (x,y), we get

$$G(x,y) = F(x,y) + F(y,x) - F(x,x)$$

Finally, differentiating gives

$$g(x,y) = \frac{\partial}{\partial x \partial y}(F(x,y) + F(y,x) - F(x,x))$$

$$= f(x,y) + f(y,x)$$