Probability, Solutions to HW3

Practice problems:

Book, 87(a) 1/3, obivously. **(b)** Let F: fake quarter and H: heads. By Bayes' rule:

$$\begin{aligned} P(F|H) &= \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|F^c)P(F^c)} \\ &= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 2/3} \end{aligned}$$

You can also think of it as choosing a side from the 3 coins HT, HT, and HH. If H has been chosen, there are 4 possibilities, 2 of which have H on the other side. Thus, the conditional probability is 1/2. Note that the observation of heads makes the belief in the 2-headed quarter fo up from 1/3 to 1/2.

Book, 90. Consider the events U: first urn and F: pick 5. Bayes' rule gives

$$P(U|F) = \frac{P(F|U)P(U)}{P(F|U)P(U) + P(F|U^c|U^c)} = \frac{\frac{1}{10} \cdot \frac{1}{2}}{\frac{1}{10} \cdot \frac{1}{2} + \frac{1}{100} \cdot \frac{1}{2}} = \frac{10}{11}$$

Book, 93. With the notation from class we get

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)} = \frac{1 \cdot 0.002}{1 \cdot 0.002 + 0.05 \cdot 0.998} = 0.0385$$

that is, a 4% chance you have it. Although this is small, note that the unconditional probability is only 0.2% so testing positive gives a 20-fold increase in the chance of having the disease.

2. Let F: flood-damaged and E: engine problems. Bayes' rule gives

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F')P(F')} = \frac{0.8 \cdot 0.05}{0.8 \cdot 0.05 + 0.10 \cdot 0.95} \approx 0.30$$

so it is less likely that flood damage caused your engine problems than that they are due to other causes.

Turn-in problems

1. Introduce the events W: something is wrong, and L: light comes on. By Bayes' rule

$$P(W|L) = \frac{P(L|W)P(W)}{P(L|W)P(W) + P(L|W^c)P(W^c)} = \frac{0.75 \cdot 0.05}{0.75 \cdot 0.05 + 0.10 \cdot 0.9} \approx 0.28$$

2. With the notation from class we have $P(D) = 0.02$, $P(H) = 0.98$, $P(+|D) = 1$ and $P(H|+) = 0.05$, which gives $P(D|+) = 0.95$. We are looking for $P(+|H)$ and by Bayes' rule:

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|H)P(H)}$$

we can view this as an equation for the unknown probability $P(+|H)$. Thus, let $x = P(+|H)$ and consider the equation

$$0.95 = \frac{0.02}{0.02 + 0.98x}$$

which has solution $x \approx 0.0011$

3(a) Let T: Carol told the truth, $L = T^c$ (Carol lied) and A: Ann said that Bob said that Carol told the truth. We want $P(T|A)$ and by Bayes' rule

$$P(T|A) = \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|L)P(L)}$$

where $P(T) = 1/3$ and $P(L) = 2/3$. For the conditional probabilities, $P(A|T)$ is the probability that Ann says that Bob claims that Carol told the truth, given that Carol told the truth. This can happen in 2 ways: both Ann and Bob tell the truth, or both Ann and Bob lies (in the latter case Bob says "Carol lied" and then Ann lies and says "Bob said that Carol told the truth"). Hence

$$P(A|T) = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9}$$

Similarly, $P(A|L)$ is the probability that Ann says that Bob claims that Carol told the truth, given that Carol lied. This can happen if Bob lies and Ann tells the truth, or if Bob tells the truth and Ann lies. Hence

$$P(A|L) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

which gives

$$P(T|A) = \frac{5/9 \cdot 1/3}{5/9 \cdot 1/3 + 4/9 \cdot 2/3} = \frac{5}{13}$$

(b) Condition on T. In analogy with the reasoning in (b), the event A (Ann said that Bob said that Carol said that Dionysus told the truth) will occur if there is an even number of liars among Ann, Bob, and Carol. Thus

$$P(A|T) = (1/3)^3 + 3 \cdot (1/3)(2/3)^2$$

where the first term is the probability that everybody tells the truth (0 liars), and the second is the probability that exactly 2 of them lie (and there is $\binom{3}{2} = 3$ ways to choose those 2). Similarly, conditioned on L, A will occur ig there is an odd number of liars so

$$P(A|L) = 3 \cdot (1/3)^2(2/3) + (2/3)^3$$

and we get

$$P(T|A) = \frac{((1/3)^3 + 3 \cdot (1/3)(2/3)^2)(1/3)}{((1/3)^3 + 3 \cdot (1/3)(2/3)^2)(1/3) + (3 \cdot (1/3)^2(2/3) + (2/3)^3)(2/3)}$$

$$= \frac{13}{41}$$

(c) For 2,3, and 4 people, respectively, we get the probabilities $1/5, 5/13, 13/41$ which rounded to 2 decimals become $0.20, 0.39, 0.32$. It seems plausible that they keep oscillating, converging to $1/3$ which is the unconditional probability of lying. This can be proved my describing the problem in terms of *Markov chains* which will be done in the Stochastic Processes class in the spring.

Extra credit problem 1: Let T be the event that the first person told the truth, L the event that he liead, and let A be the event that person $n+1$ says that person n says that...and so on. We have

$$P(T|A) = \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|L)P(L)} = \frac{P(A|T)}{P(A|T) + 2P(A|L)}$$

By the reasoning above, we realize that the conditional probabilities depend on whether the number of liars is odd or even. For example, conditioned on T, the statement A will occur if the number of liars is even, that is of the form $2j$ for $j = 0, ..., n/2$. The probability to get $2j$ *particular* liars (and thus $n-2j$ truth-tellers) is $(2/3)^{2j}(1/3)^{n-2j}$ and as we can choose the $2j$ liars in $\binom{n}{2j}$ ways, we get

$$P(A|T) = \binom{n}{2j}(2/3)^{2j}(1/3)^{n-2j} = \binom{n}{2j}(1/3)^n 2^{2j}$$

Similarly, conditioned on L, A will occur if the number of liars is odd and we get

$$P(A|L) == \binom{n}{2j+1}(2/3)^{2j+1}(1/3)^{n-(2j+1)} = \binom{n}{2j+1}(1/3)^n 2^{2j+1}$$

for $j = 0, ..., n/2 - 1$. Plugging $P(A|T)$ and $PA(|L)$ into the formula above, noting that $(1/3)^n$ cancels, gives the result.

Extra credit problem 2: Use induction over n. First let $n = 2$. Then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

since $P(A_1 \cap A_2) \geq 0$. Thus, the statement is true for $n = 2$. Now assume it is true for n and show that it is then true for $n+1$. We have

$$\begin{aligned} P\left(\bigcup_{k=1}^{n+1} A_k\right) &= P\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right) \\ &\leq P\left(\bigcup_{k=1}^{n} A_k\right) + P(A_{n+1}) \end{aligned}$$

by what we just proved for 2 events (here applied to $\bigcup_{k=1}^{n} A_k$ and A_{k+1}). By the induction hypothesis,

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k)$$

and the proof is complete.