Probability, Solutions to HW4

2. For any x, $F(x) = P(X \le x)$. Moreover, the probability $P(X = x)$ is the size of the jump in F at the point x. Thus

(a) $P(X = 1) = 1/4$ **(b)** $P(X = 2) = 1/2$ **(c)** $P(X = 2.5) = 0$ **(d)** $P(X \le 2.5) = 3/4$

Note that the cdf F tells us that X can take on the values 1, 2, and 3 with probabilities $1/4, 1/2$, and $1/4$, respectively.

4(a) To find c, sum over k and set the sum equal to 1:

$$1 = c\sum_{k=1}^{3} k = 6c$$

which gives $c = 1/6$ and $p(k) = k/6, k = 1, 2, 3$.

(b) The cdf F has $F(1) = p(1) = 1/6, F(2) = p(1) + p(2) = 1/2, F(3) = p(1) + p(2) + p(3) = 1$, is constant between possible values of X, and is right-continuous. Thus

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/6 & \text{for } 1 \le x < 2 \\ 1/2 & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

(c) $P(X \le 2) = F(2) = 1/2$.

(d) $P(X > 1) = 1 - F(1) = 5/6$.

5(a) To find c, sum over k and set the sum equal to 1:

$$1 = c\sum_{k=0}^{\infty} \frac{1}{2^k} = 2c$$

which gives $c = 1/2$ and $p(k) = 1/2^{k+1},\ k = 0, 1, ...$

(b) $P(X > 0) = 1 - P(X = 0) = 1 - 1/2 = 1/2$.

(c) Let E be the set of even numbers. Then

$$P(X \in E) = \sum_{k=0}^{\infty} p(2k) = \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{2}{3}$$

8(a) The range is $1, 2, 3, ...$ and the probability to get the ace of spades in any given draw is $1/52$. To get $X = k$ we must start with $k - 1$ failures and we get

$$f(k) = \left(\frac{51}{52}\right)^{k-1} \frac{1}{52},\ k = 1, 2, ...$$

similarly to the example we did in class where we wait for the first 6 in repeated rolls of a die.

(b) The range is $1, 2, ..., 52$. The probability that the ace of spades appears in the first trial is $1/52$, hence $f(1) = 1/52$. That it appears in the second trial means that we must draw something else in the first trial, which has probability $51/52$, and then draw the ace of spades from the remaining 51 cards which has probability $1/51$. Hence $f(2) = 51/52 \cdot 1/51 = 1/52$ (formally we have used the formula $P(A \cap B) = P(B|A)P(A)$ here). Continuing in this way we get

$$f(k) = \frac{1}{52},\ k = 1, 2, ..., 52$$

that is, a uniform distribution. This is intuitively clear because the ace of spades is equally likely to be in any of the 52 positions. Do not confuse conditional and unconditional probabilities here: The (unconditional) probability that the ace of spades appears in the 51st trial is $1/52$ but the *conditional* probability that it appears in the 51st trial *given that it has not appeared before* is $1/2$.

12. Counterexample: If $X \sim$ unif $[0, 1/2]$, $f(x) = 2$ for all x in the range of X.

13(a) To find c, integrate x over the range of X and set the integral equal to 1:

$$1 = \int_0^1 cx^2 dx = \frac{c}{3}$$

which gives $c = 3$.

(b) The cdf is

$$F(x) = \int_0^x f(t)dt = \int_0^x 3t^2 dt = x^3,\ 0 \le x \le 1$$

which gives $P(X > 0.5) = 1 - F(0.5) = 1 - 0.5^3 = 0.875$.

(c) The cdf of Y is

$$F_Y(x) = P(Y \le x) = P(\sqrt{X} \le x) = P(X \le x^2) = F_X(x^2) = x^6$$

and hence Y has pdf $f_Y(x) = F_Y'(x) = 6x^5, 0 \le x \le 1$.

14. For each function, check whether it is (1) ≥ 0 and (2) has $\int f(x)dx = 1$. In **(b)**, (1) fails for x between 0 and 1 so it is not a possible pdf. In **(c)**, (1) and (2) hold so it is a possible pdf and the cdf is

$$F(x) = \int_{-1}^x dt = x + 1, -1 \le x \le 0$$

Note that this is a uniform distribution on $[-1, 0]$.

16(a) Integrate and set equal to one:

$$1 = \int_a^\infty \frac{dx}{x^3} = \left[-\frac{1}{2x^2}\right]_a^\infty = \frac{1}{2a^2}$$

which gives $a = 1/\sqrt{2}$.

(b)

$$P(X > 3) = \int_3^\infty \frac{dx}{x^3} = \left[-\frac{1}{2x^2}\right]_3^\infty = \frac{1}{18}$$

(c)

$$P(X > x) = \int_x^\infty \frac{dt}{t^3} = \frac{1}{2x^2} = \frac{1}{4}$$

gives $x = \sqrt{2}$.

18(a) Clearly $f \geq 0$ and since

$$\int_0^\infty f(x)dx = \int_0^1 \frac{1}{2}dx + \int_1^\infty \frac{1}{2x^2}dx$$

$$= \frac{1}{2} + \left[-\frac{1}{2x}\right]_1^\infty = \frac{1}{2} + \frac{1}{2} = 1$$

it is a possible pdf.

(b) For the cdf, consider the two cases $0 \leq x \leq 1$ and $x > 1$. For $0 \leq x \leq 1$:

$$F(x) = \int_0^x \frac{1}{2}dx = \frac{x}{2}, \quad 0 \leq x \leq 1$$

and for $x > 1$:

$$F(x) = \int_0^x f(t)dt = \int_0^1 \frac{1}{2}dx + \int_1^x \frac{1}{2t^2}dt$$

$$= \frac{1}{2} + \left[-\frac{1}{2t}\right]_1^x = 1 - \frac{1}{2x}, \quad x > 1$$

Note that this is a continuous function with $F(1) = 1/2$. Also note that F is increasing and has $F(\infty) = 1$.

(c) Start with the cdf of Y:

$$F_Y(x) = P(Y \leq x) = P\left(\frac{1}{X} \leq x\right) = P\left(X \geq \frac{1}{x}\right) = 1 - F_X\left(\frac{1}{x}\right)$$

Now recall F_X from above and note that if $0 < x \leq 1$, we have $\frac{1}{x} \geq 1$ and hence

$$F_Y(x) = 1 - \left(1 - \frac{1}{2(1/x)}\right) = \frac{x}{2}$$

and if $x > 1$ we have $\frac{1}{x} < 1$ and

$$F_Y(x) = 1 - \frac{1/x}{2} = 1 - \frac{1}{2x}$$

that is, Y has the same distribution as X.

19. Let $Y = 1 - X$ and start with the cdf of Y:

$$F_Y(x) = P(Y \le x) = P(1 - X \le x) = P(X \ge 1 - x) = 1 - F_X(1 - x)$$

and since $F_X(t) = t$, we get $F_Y(x) = x$, that is, Y is also uniform on $[0, 1]$.

22. As always, start with the cdf:

$$F_Y(x) = P(Y \le x) = P(\sqrt{X} \le x) = P(X \le x^2) = F_X(x^2)$$

We need the cdf of X:

$$F_X(x) = \int_0^x e^{-t} dt = 1 - e^{-x}$$

which gives

$$F_Y(x) = 1 - e^{-x^2}$$

and the pdf

$$f_Y'(x) = F_Y'(x) = 2xe^{-x^2}$$

2. We have $A = \pi R^2$ which gives $R = \sqrt{\frac{A}{\pi}}$ where $A \sim$ unif $[0, \pi]$. The range of R is $[0, 1]$. Take an x in the range and start with the cdf of R:

$$F_R(x) = P(R \le x) = P(\sqrt{\frac{A}{\pi}} \le x) = P(A \le \pi x^2) = F_A(\pi x^2), \quad 0 \le x \le 1$$

As $A \sim$ unif $[0, \pi]$, the cdf of A is $F_A(t) = \frac{t}{\pi}, \quad 0 \le t \le \pi$ which gives

$$F_R(x) = \frac{\pi x^2}{\pi} = x^2, \quad 0 \le x \le 1$$

Finally, differentiate to get the pdf:

$$f_R(x) = F_R^{'}(x) = 2x, \quad 0 \le x \le 1$$

1. The range of A is $(0, 4]$. Take an x in this range and start with the cdf of A:

$$F_A(x) = P(A \le x) = P(X^2 \le x) = P(X \le \sqrt{x}) = \frac{\sqrt{x}}{2}$$

since the cdf of X is $F(t) = t/2, 0 \le t \le 2$ and we have $t = \sqrt{x}$. We get the pdf

$$f_A(x) = F_A'(x) = \frac{1}{4\sqrt{x}}, \quad 0 < x \le 4$$