Probability, solutions to HW5

Practice problems.

1(a) The range is $\{-1, a\}$ where a is the payout and the pmf is $f(-1) = 35/38,\ f(a) = 3/38$. The expected value satisfies

$$E[X] = (-1)\frac{35}{38} + a\frac{3}{38} = -\frac{2}{38}$$

which we solve for a to get $a = 11$.

(b) For the variance we need

$$E[X^2] = (-1)^2\frac{35}{38} + 11^2\frac{3}{38} = \frac{398}{38}$$

which gives variance $\text{Var}[V] = E[V^2] - (E[V])^2 \approx 10.5$. It is smaller than the variance of a straight bet and larger than the variance of an odd bet.

Book, 38. The mean is

$$E[X] = \int_0^1 xf(x)dx = \int_0^1 3x^3 dx = \frac{3}{4}$$

For the variance we need

$$E[X^2] = \int_0^1 x^2 f(x)dx = \int_0^1 3x^4 dx = \frac{3}{5}$$

which gives variance $\text{Var}[X] = 3/5 - (3/4)^2 = 3/80$.

(b) Use Proposition 2.12 with $g(x) = \sqrt{x}$ to get

$$E[Y] = \int_0^1 \sqrt{x} f(x)dx = \frac{6}{7}$$

For the variance we need

$$E[Y^2] = \int_0^1 xf(x)dx = \frac{3}{4}$$

which gives variance $\text{Var}[Y] = 3/4 - (6/7)^2 = 3/196$.

Book, 42. We have $V = X^3$. The mean is

$$E[V] = \int_0^2 x^3 f(x)dx = \left[\frac{x^4}{4} \cdot \frac{1}{2}\right]_0^2 = 2$$

For the variance we need

$$E[V^2] = \int_0^2 x^6 f_X(x)dx = 64/7$$

which gives variance $\text{Var}[V] = E[V^2] - (E[V])^2 = 36/7$.

43. $E[X] = \int_0^1 xf(x)dx = \int_0^1 2x^2 dx = 2/3$

$\text{Var}[X] = E[X^2] - (E[X])^2$ where $E[X] = 2/3$ and $E[X^2] = \int_0^1 x^2 f(x)dx = \int_0^1 2x^3 dx = 1/2$; thus $\text{Var}[X] = 1/2 - (2/3)^2 = 1/18$.

(b) $E[4\pi X^3/3] = \int_0^1 \frac{4\pi}{3} x^3 f(x)dx = \frac{4\pi}{3} \int_0^1 2x^4 dx = \frac{8\pi}{15}$.

Turn-in problems.

1(a) Solve $(-1) \cdot 33/38 + a \cdot 5/38 = -2/38$ for a to get the payout $a = 6.2$.
(b) $-3/38 \approx -0.08$, an expected loss of 8 cents per dollar.

2(a) The range of X is $(-\infty, \infty)$. The point X and angle ϕ relate as $X = \tan\phi$ and as the tangent function is invertible on $(-\pi/2, \pi/2)$ we get

$$F_X(x) = P(X \leq x) = P(\tan\phi \leq x) = P(\phi \leq \arctan(x))$$

$$= F_\phi(\arctan(x)) = \frac{\arctan(x)}{\pi} + \frac{1}{2}$$

since ϕ has cdf

$$F_\phi(t) = \frac{t + \pi/2}{\pi}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

Differentiate to get the pdf:

$$f_X(x) = F'_X(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in R$$

(b) By definition:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$$

$$= \frac{1}{\pi} \left[\frac{1}{2} \log(1+x^2) \right]_{-\infty}^{\infty}$$

which is of the form "$\infty - \infty$" and is thus not defined. We conclude that X has no expected value.