# Probability, solutions to HW6

**Practice problems.**

**50(a)** Binomial with $n = 10$ and $p = 0.8$.
**(b)** Not binomial as trials are not independent (one rainy day makes another more likely)
**(c)** Not binomial as $p$ changes between months.
**(d)** Binomial with $n = 10$ and $p = 0.2$.

**51.** First find $n$ and $p$. By the formulas for mean and variance, we have $np = 1$ and $np(1-p) = 0.9$. We get $1 - p = 0.9$, that is $p = 0.1$, and $n = 1/p = 10$. Finally

$$P(X > 0) = 1 - P(X = 0) = 1 - (1-p)^n = 1 - 0.9^{10} \approx 0.65$$

**53(a)** Since $X \sim \text{bin}(6, 1/6)$ and $Y \sim \text{bin}(12, 1/6)$, we have $E[X] = 1$ and $E[Y] = 2$
**(b)** $P(X > E[X]) = P(X > 1) = 1 - (P(X = 0) + P(X = 1)) \approx 0.26$, $P(Y > E[Y]) = P(Y > 2) = 1 - (P(Y = 0) + P(Y = 1) + P(Y = 2)) \approx 0.32$. Note that the probability is closer to 0.5 in the latter case. This trend continues, the probabilities $P(X > E[X])$ approach 0.5 as $n$ increases and the binomial distribution becomes more and more symmetric around its mean.

**54(a)** Let $A$ be the event that the first flip shows heads. We then want the conditional probability $P(X = k|A)$ and We can write $X = 1 + Y$ where where $Y \sim \text{bin}(n-1, p)$. Thus, $P(X = k|A) = P(Y = k-1) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$.
**(b)** Here instead, $X = Y$ and we need $P(X = k|A^c) = P(Y = k)$.
**(c)** With $X \sim \text{bin}(n, p)$, we have

$$P(X = k|X > 0) = \frac{P(X = k, X > 0)}{P(X > 0)} = \frac{P(X = k)}{P(X > 0)} = \frac{1}{1 - (1/2)^n} \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 1, 2, ..., n$. Note that there is a difference between this expression and the one in **(a)**; to condition on $A$: heads in first flip, is not the same as conditioning on $X > 0$ (the latter event is larger).

for $k = 1, 2, ..., n$.

**59.** The distribution of $X$ is $\text{bin}(n, p)$ where $p$ is the probability that at least 5 flips is required. Hence, $p = P(Y \geq 5)$ where $Y \sim$geom$(1/2)$ and since $Y \geq 5$ means that the first 4 flips give tails, we get $P(Y \geq 5) = (1/2)^4 = 1/16$. Hence, $P(X = 0) = (15/16)^n$ and $E[X] = n/16$.

**71.** No. Let $X$ be a lifetime. The data suggest that $P(X > 1) \approx 0.80$ and $P(X > 2|X > 1) \approx 30/80 = 0.375$ but an exponential distribution would have the two probabilities equal. We do expect some random variation but the difference between 0.80 and 0.375 is probably too large to believe in an exponential distribution. These lightbulbs seem to get worse with age.

**72.** The cdf of $Y$ is

$$F_Y(x) = P(Y \leq x) = P(\lambda X \leq x) = F_X(x/\lambda) = 1 - e^{-\lambda x/\lambda} = 1 - e^{-x}$$

which is the cdf of the exp(1) distribution.

**73.** Use the fact that $\lambda = \log 2/h$.

**74.** First note that $N = k$ if and only if $k - 1 \leq T \leq k$ for $k = 1, 2, ...$ and since $T \sim \exp(\lambda)$, its cdf is $F_T(x) = 1 - e^{-\lambda x}$ and we get

$$P(N = k) = F(k) - F(k-1) = e^{-\lambda(k-1)} - e^{-\lambda k} = e^{-\lambda(k-1)}(1 - e^{-\lambda})$$

which is precisely the pmf of a geometric distribution with $p = 1 - e^{-\lambda}$. If $T$ is a lifetime of some component, $N$ could for example be the number of days it functions.

**109(a)** The cdf is

$$F(t) = \int_0^t 2x dx = t^2$$

which gives failure rate function

$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{2t}{1 - t^2}m, \quad 0 \leq t < 1$$

Note that $r(t) \to \infty$ as $t \to 1$ because the random variable is bounded above by 1.

**110(a)** First note that

$$\int_0^t r(s)ds = \int_0^t \frac{ds}{1+s} = \log(1+t)$$

which gives cdf

$$F(t) = 1 - exp(-\log(1+t)) = 1 - \frac{1}{1+t}, \quad t \geq 0$$

**2(a)** Assume that meteorites hit according to a Poisson process with rate $\lambda$ (hits per millennium). Let $X$ be the number of hits within the next 1,000 years. Then $X \sim \text{Poi}(1)$ and we get $P(X > 0) = 1 - P(X = 0) = 1 - e^1 = 0.63$.

**(b)** $P(X \geq 2) = 1 - P(X \leq 1) = 1 - (P(X = 0) + P(X = 1)) = 1 - (e^{-1} + e^{-1}) = 0.26$

**(c)** Here $X \sim \text{Poi}(0.1)$ and $P(X = 0) = e^{-0.1} = 0.9$.