Probability, solutions to HW7

77(a) $P(X \leq 220) = \Phi((220 - 200)/10) = \Phi(2) = 0.98$
(b) $P(X \leq 190) = \Phi((190 - 200)/10) = \Phi(-1) = 1 - \Phi(1) = 0.16$
(c) $P(X > 185) = \Phi(1.5) = 0.93$
(d) $P(X > 205) = 1 - \Phi(0.5) = 0.31$
(e) $P(190 \leq X \leq 210) = \Phi(1) - \Phi(-1) = 0.68$
(f) $P(180 \leq X \leq 210) = \Phi(1) - \Phi(-2) = 0.81$

78. Done in class.

79. We have $X \sim$N$(70, 3^2)$ and need to find a and b such that $P(X \leq a) = 0.10$ and $P(X \leq b) = 0.90$. We get $P(X \leq a) = \Phi((a - 70)/3) = 1 - \Phi(-(a - 70)/3) = 0.10$ which gives $\Phi(-(a - 70)/3) = 0.90$ which gives $-(a - 70)/3 = 1.28$ and $a = 70 - 1.28 \cdot 3 = 66$. Further, $P(X \leq b) = \Phi((b - 70)/3) = 0.90$ gives $b = 70 + 1.28 \cdot 3 = 74$.

80. $Z = (\mu + c\sigma - \mu)/\sigma = c$. Thus, the Z-score measures by how many standard deviations X differs from μ. If X is smaller than μ, the Z-score is negative ($c < 0$).

84(a) $T \sim N(t + 0.1, 0.01)$.
(b) $T \sim N(11.6, 0.01)$ and $P(T \leq 11.5) = \Phi((11.5 - 11.6)/0.1) = \Phi(-1) = 1 - \Phi(1) = 0.16$.
(c) $P(t - 0.05 \leq T \leq t + 0.05) = \Phi((t + 0.05 - (t + 0.1))/0.1) - \Phi((t - 0.05) - (t + 0.1))/0.1) = \Phi(-0.5) - \Phi(-1.5) = 0.24$.

85. In each case, start with the cdf and differentiate to get the pdf, recalling that $\Phi'(x) = \varphi(x)$.

(a) Range: R. $P(-X \leq x) = P(X \geq -x) = 1 - \Phi(-x) = \Phi(x)$ which has derivative $\varphi(x)$.
(b) Range: $[0, \infty)$. $P(|X| \leq x) = P(-x \leq X \leq x) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1$ which has derivative $2\varphi(x)$.
(d) Range: $(0, \infty)$. $P(e^X \leq x) = P(X \leq \log x) = \Phi(\log x)$ which has derivative $\varphi(\log x)/x$.

81. The z-scores are

$$Z_A = \frac{24-20}{2} = 2$$

and

$$Z_B = \frac{48-40}{8} = 1$$

so the 48-pound A-fish is more extreme.

82. Her Z-score on the first test is $(80-70)/10 = 1$ and to get the same Z-score on the second test, she needs to score 180 points.

83. We get the correct answer if $Y = n + X$ is rounded to n (and not to $n-1$ or $n+1$ or some other integer). This occurs when X is between -0.5 and 0.5 which has probability $P(-0.5 \leq X \leq 0.5) = \Phi(0.5/0.43) - \Phi(-0.5/0.43) = \Phi(1.16) - \Phi(-1.16) = 2\Phi(1.16) - 1 = 0.75$.

85(c) Range: $[0, \infty)$. $P(X^2 \leq x) = P(-\sqrt{x} \geq X \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = 2\Phi(\sqrt{x}) - 1$ which has derivative $2\varphi(\sqrt{x})/2\sqrt{x} = \varphi(\sqrt{x})/\sqrt{x}$