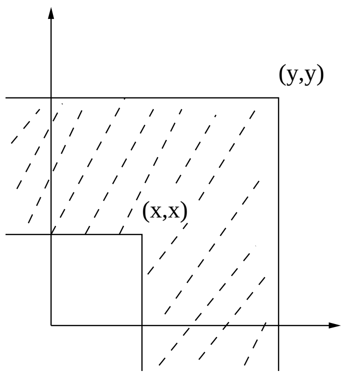
Probability, solutions to HW8

Practice problems

1. Remember that $F(x, y)$ is the integral of the joint pdf over the region $(-\infty, x] \times (-\infty, y]$, then add and subtract to get the answer.

2. This one:



6. Recall that $F_X(x) = F(x, \infty)$ and $F_Y(y) = F(\infty, y)$. If the range is unbounded, we interpret these as limits as x and y go to infinity; if the range is bounded we insert the upper bound of the range. For example:

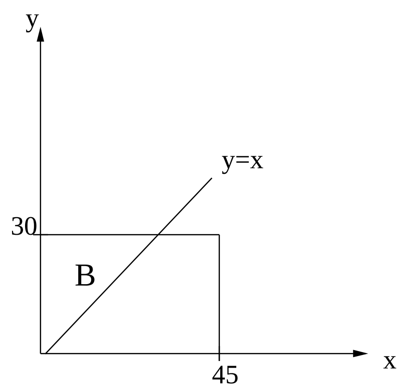
(a) $F_X(x) = F(x, \infty) = \lim_{y\to\infty}(1 - e^{-x} - e^{-y} + e^{-(x+y)}) = 1 - e^{-x}$. Note that this is an exponential distribution with $\lambda = 1$ and that Y has the same distribution and that X and Y are independent since the joint cdf equals the product of the marginal cdf's.

(c) $F_X(x) = F(x, \infty) = F(x, 1) = 2x, 0 \leq x \leq 1/2$ and $F_Y(y) = F(\infty, y) = F(1/2, y) = y, 0 \leq y \leq 1$.

34(a) Let X be Billy Bob's arrival and Y be Adam's arrival, in minutes after 12:30. Then $X \sim$ unif $[0, 45]$ and $Y \sim$ unif $[0, 30]$. Consider the pair (X, Y). By independence, the joint pdf is $f(x, y) = f_X(x)f_Y(y) = 1/1350, \quad 0 \leq x \leq 45, 0 \leq y \leq 30$. We get

$$P(\text{Billy Bob arrives first}) = P(X < Y) = P((X, Y) \in B)$$

where B is the region below.

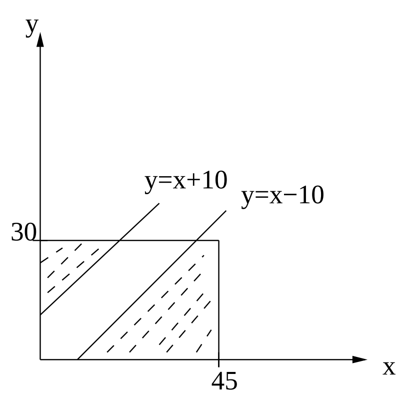


We get

$$P(X<Y)=\int\int_B f(x,y)dxdy=\frac{\text{area of } B}{1350}=\frac{450}{1350}=\frac{1}{3}.$$

(b) The region B is now the shaded region below and

$$P((X,Y)\in B)=800/1350=16/27\approx 0.59$$



36. Check if $f(x,y)=f_X(x)f_Y(y)$. For example, in **(a)** we get

$$f_X(x)=\int_0^1 f(x,y)dy=4x\int_0^1 ydy=2x,\ 0\le x\le 1$$

and $f_Y(y)=2y,\ 0\le y\le 1$ and hence $f(x,y)=f_X(x)f_Y(y)$ so X and Y are independent.

Turn-in problems

16(a)

$$1 = \int\int f(x,y)dxdy = c\int_{x=0}^{1}\int_{y=0}^{x}(x-y)dydx$$
$$= c\int_{x=0}^{1}\left[xy-\frac{y^2}{2}\right]_{y=0}^{x}dx$$
$$= c\int_0^1 \frac{x^2}{2}dx = \frac{c}{6}$$

which gives $c = 6$.

(b)

$$P(X > \frac{1}{2}, Y \le \frac{1}{2}) = \int_{x=1/2}^{1}\int_{y=0}^{1/2} 6(x-y)dxdy = \frac{3}{4}$$

(c)

$$P(X \le 2Y) = \int_{x=1/2}^{1}\int_{y=x/2}^{x} 6(x-y)dydx = \frac{1}{4}$$

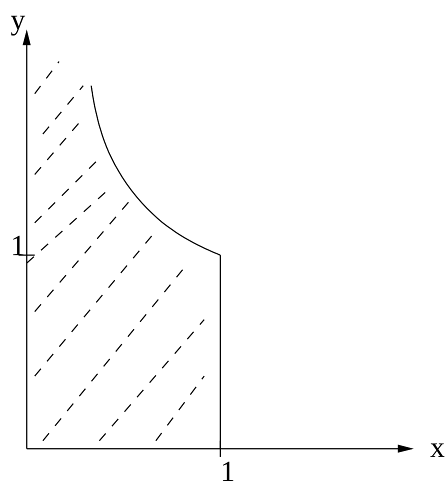
(d)

$$f_X(x) = \int f(x,y)dy = \int_0^x 6(x-y)dy = 6\left[xy-\frac{y^2}{2}\right]_0^x = 3x^2,\ 0 \le x \le 1$$

$$f_Y(y) = \int f(x,y)dx = \int_y^1 6(x-y)dx = 6\left[\frac{x^2}{2}-xy\right]_y^2 = 3(y-1)^2,\ 0 \le y \le 1$$

32(a) $f(x,y) = f_X(x)f_Y(y|x)$ where $f_X(x) = 1, 0 \le x \le 1$ and $f_Y(y|x) = \dfrac{1}{1/x} = x, 0 \le y \le 1/x$. Hence, $f(x,y) = x, \quad 0 \le x \le 1, 0 \le y \le 1/x$ which is the shaded region below.
(b) For $0 \le y \le 1:$

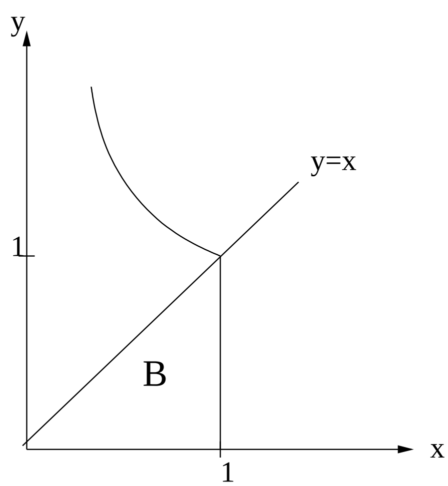
$$f_Y(y) = \int_0^1 xdx = \frac{1}{2}$$



and for $y \geq 1$:

$$f_Y(y) = \int_0^{1/y} x dx = \frac{1}{2y^2}$$

(c) $P(X > Y) = P((X, Y) \in B) = \int\int_B f(x,y) dx dy$ where B is the triangular region below



We get

$$P(X > Y) = \int_{x=0}^{1} \int_{y=0}^{x} x dy dx = \int_{0}^{1} x^2 dx = 1/3$$

2(a) Since (X, Y) is uniform on the triangle, $f(x, y)$ must be constant and since the area of the triangle is $1/2$, we get $f(x, y) = 2$.

(b) The range of X is $[0, 1]$. For fixed x in this range we get

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = 2 \int_{0}^{x} dy = 2x, \ 0 \le x \le 1$$

The range of Y is also $[0, 1]$. For fixed y in this range we get

$$f_Y(y) = 2 \int_{y}^{1} dx = 2(1 - y), \ 0 \le y \le 1$$

(Think about why these pdf's make intuitive sense.)

(c) By definition

$$f_Y(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}, \quad 0 \le y \le x$$

that is, $Y|X = x \sim$unif $[0, x]$. For $X|Y = y$ note that

$$f_X(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{2}{2(1 - y)} = \frac{1}{1 - y}, \quad y \le x \le 1$$

that is, $X|Y = y \sim$unif $[y, 1]$.

3. Let

$$c = \frac{1}{\text{area of } B}$$

so that $f(x, y) = c, \ (x, y) \in B$. The conditional pdf of Y given $X = x$ is

$$f_Y(y|x) = \frac{c}{f_X(x)}$$

which is constant as a function of its argument y, thus it is a uniform distribution.