Probability, solutions to HW9

85. We have $\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$. First note that $XY = UV$ and hence $E[XY] = E[UV] = E[U]E[V] = 1/4$. Next, we get

$$E[X] = E[\min(U,V)] = \int_0^1 \int_0^1 \min(u,v) f(u,v)\,du\,dv$$

where $\min(u,v) = u$ if $u \le v$ and $\min(u,v) = v$ if $u \ge v$. Hence

$$E[X] = \int_{v=0}^1 \left(\int_0^v u\,du + \int_v^1 v\,du \right) dv = \frac{1}{3}$$

and, similarly, $E[Y] = 2/3$. We get

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{36}$$

The covariance is positive because if the minimum is large, the maximum also tends to be large.

95. For independence, check if $f(x,y) = f_X(x) f_Y(y)$ (or if $f_Y(y|x) = f_Y(y)$). For uncorrelatedness, check if $\mathrm{Cov}[X,Y] = 0$ which will happen if $E[XY] = E[X]E[Y]$. For (b), (c), and (d), note that X and Y are clearly dependent, but there are no linear relationships.

96. If $X \sim$ unif $[0,1]$, we know that $E[X] = 1/2$ and $\mathrm{Var}[X] = 1/12$, and the variance formula gives

$$E[X^2] = \mathrm{Var}[X] + E[X]^2 = \frac{1}{3}$$

The covariance of A and C is

$$\mathrm{Cov}[A,C] = E[AC] - E[A]E[C]$$

$$= E[XY(2X+2Y)] - E[XY]E[2X+2Y]$$

$$= 2E[X^2]E[Y] + 2E[X]E[Y^2] - E[X]E[Y] \cdot 2(E[X] + E[Y]$$

where we have used the fact that X and Y (and hence X and Y^2 etc) are independent. Plugging in the values from above gives $\mathrm{Cov}[A,C] = 1/6$. The variances are

$$\mathrm{Var}[A] = E[A^2] - E[A]^2$$

$$= E[X^2]E[Y^2] - E[X]^2E[Y]^2 = \frac{7}{144}$$

and

$$\mathrm{Var}[C] = 4(\mathrm{Var}[X] + \mathrm{Var}[Y]) = \frac{2}{3}$$

which finally gives

$$\rho(A, C) = \frac{1/6}{\sqrt{7/144 * 2/3}} = \sqrt{\frac{6}{7}} \approx 0.93$$

97. The covariance of S and T is

$$\mathrm{Cov}[S, T] = E[ST] - E[S]E[T]$$

$$= E[(X + Y)XY] - E[X + Y]E[XY]$$

$$= E[X^2]E[Y] + E[X]E[Y^2] - E[X]^2E[Y] - E[X]E[Y]^2$$

$$= (E[X^2] - E[X]^2)E[Y] + (E[Y^2] - E[Y]^2)E[X]$$

$$= \sigma^2(E[X] + E[Y])$$

which eqeuals 0 if and only if $E[X] + E[Y] = 0$.