Probabilistic Models, Solutions 1

Practice problems

1. Both (c) and (e) can be used. Note that (e) is more detailed as it takes into account the entire sequence of heads and tails. For example, the outcome “2” in (c) corresponds to the event \{HHT, HTH, THH\} in (e).

2. Sample space \(\Omega = \{H, T, E\}\) where \(P(E) = 0.1\) and \(P(H) = 2P(T)\). As \(P(H) + P(T) + P(E) = 1\), we get the equation \(3P(T) = 0.9\) which gives \(P(T) = 0.3\) and \(P(H) = 0.6\).

3. 

![Venn diagram with A, B, and their intersections]

(a) \(P(B) = 0.4\)
(b,c) \(P(A \cap B^c) = 0.1\)
(d) \(P(A^c) = 0.7\)
(e) \(P(B^c) = 0.6\)
(f) \(P(A^c \cap B^c) = 0.5\) (which also equals \(P((A \cup B)')\))
(g) \(P(A \cup B^c) = 0.1 + 0.2 + 0.5 = 0.8\) (anything that is in \(A\) or outside \(B\))
(h) \(P((A \cap B) \cup (A^c \cap B^c)) = 0.2 + 0.5 = 0.7\) (anything that is in both \(A\) and \(B\) or in neither of them)

4(a) \(26^3 \cdot 10^2 = 1,757,600\)
(b) \(26 \cdot 25 \cdot 24 \cdot 10^2 = 1,560,000\)
(c) \(26^3 \cdot 10 \cdot 9 = 1,581,840\)
(d) \(26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 = 1,404,000\)

(e) Number of ways to choose 3 letters in order: \(26 \cdot 25 \cdot 24\)
Number of ways to choose 2 digits in order: 10 · 9

Number of ways to choose positions for the 2 digits: \( \binom{5}{2} = 10 \)

Total number of ways: \( \binom{5}{2} \cdot 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 = 14,040,000 \)

5. If \( A \) occurs, \( B \) must also occur, hence \( P(B|A) = 1 \). If \( B \) occurs, 4 out of 5 equally likely outcomes are even, hence \( P(A|B) = \frac{3}{5} \).

6(a) If the first roll gives 5, the second must give 3 to get sum 8, hence \( P(E|F) = \frac{1}{6} \). (b) If the sum equals 8, we have one of the 5 equally likely outcomes \((2,6),(3,5),(4,4),(5,3),(6,2)\) and hence \( P(F|E) = \frac{1}{5} \).

7(a) \( P(A \cap B) = P(A)P(B) = \frac{1}{8}, P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/4 + 1/2 - 1/8 = 5/8 \)

(b) \( P(A \cap B) = 0 \) (since \( A \cap B = \emptyset \)), \( P(A \cup B) = P(A) + P(B) = 3/4 \)

Turn-in problems

1(a) \( \frac{1}{365} \) (b) \( \binom{50}{2} = \frac{50 \cdot 49}{2} = 1225. \)

2. The number of ways to get the distribution 5–4–3–1 for a fixed suit order is \( \binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1} \). There are 4! different suit orders (or, if you wish, \( \binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} \)) and we get

\[
P(5–4–3–1) = \frac{4!\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} \approx 0.13
\]

and hence 5–4–3–1 is more likely than 4–3–3–3.

3. Let \( A \) and \( B \) be the events that he wins the first and second elections, respectively. Then

(a) \( P(A \cap B) = P(A)P(B|A) = 0.60 \cdot 0.75 = 0.45 \)

(b) \( P(A \cap B^c) = P(A)P(B^c|A) = 0.60 \cdot 0.25 = 0.15 \)

4(a) \( P(A \cap B \cap C) = 0 \) but \( P(A)P(B)P(C) = 1/8 \). Note that the events
are pairwise independent.

(b) For example, $A$ and $C$ are not independent since $P(A) = 1/2$ but $P(A|C) = 1$. Note that $P(A \cap B \cap C) = P(A)P(B)P(C) = 1/36$. 