Probabilistic Models, Solutions 2

Practice problems:

1. Introduce the events $E$: error and $L$: light comes on. By LTP

$$P(L) = P(L|E)P(E) + P(L|E')P(E') = 0.75 \cdot 0.05 + 0.10 \cdot 0.9 \approx 0.13$$

2(a) The range is 1, 2, 3, ... and the probability to get the ace of spades in any given draw is 1/52. To get $X = k$ we must start with $k - 1$ failures and we get

$$f(k) = \left(\frac{51}{52}\right)^{k-1} \frac{1}{52}, \quad k = 1, 2, ...$$

(a geometric distribution with success probability $p = 1/52$).

(b) The range is 1, 2, ..., 52. The probability that the ace of spades appears in the first trial is 1/52, hence $f(1) = 1/52$. That it appears in the second trial means that we must draw something else in the first trial, which has probability 51/52, and then draw the ace of spades from the remaining 51 cards which has probability 1/51. Hence $f(2) = 51/52 \cdot 1/51 = 1/52$ (formally we have used the formula $P(A \cap B) = P(B|A)P(A)$ here). Continuing in this way we get

$$f(k) = \frac{1}{52}, \quad k = 1, 2, ..., 52$$

that is, a uniform distribution. This is intuitively clear because the ace of spades is equally likely to be in any of the 52 positions. Do not confuse conditional and unconditional probabilities here: The (unconditional) probability that the ace of spades appears in the 51st trial is 1/52 but the conditional probability that it appears in the 51st trial given that it has not appeared before is 1/2.

3. Let $x$ denote “not 6” and let $X$ denote your gain. The possible outcome of $X$ and the corresponding sequences of 3 dice are

$X = -1$: $xxx$, probability $f(0) = (5/6)^3 = 125/216$

$X = 1, 6xx, x6x, xx6$, probability $f(1) = 3 \cdot (5/6)^2 \cdot (1/6) = 75/216$
\[ X = 2, \ 6x6, \ x66, \ \text{probability} \ f(2) = 3 \cdot \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right)^2 = \frac{15}{216} \]
\[ X = 3, \ x66, \ \text{probability} \ f(3) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \]

which gives expected gain

\[
E[X] = (-1) \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = -\frac{17}{216} \approx -0.079
\]

Note that \(X\) has an “almost binomial” distribution with -1 instead of 0.

4(a) Binomial with \(n = 10\) and \(p = 0.8\).
(b) Not binomial as trials are not independent (one rainy day makes another more likely)
(c) Not binomial as \(p\) changes between months.
(d) Binomial with \(n = 10\) and \(p = 0.2\).

**Turn-in problems**

1(a) Call the payout \(a\). We get the equation

\[
-\frac{2}{38} = (-1) \frac{36}{38} + a \frac{2}{38}
\]

which gives \(2a - 36 = -2\) which gives \(a = 17\).

(b) \(E[X] = (-1) \frac{33}{38} + 6 \cdot \frac{5}{38} = -\frac{3}{38} \approx -0.08\)

(c) Denote the payout by \(a\) and solve the equation

\[
-\frac{2}{38} = (-1) \frac{33}{38} + a \frac{5}{38}
\]

to get \(a = 31/5 = 6.2\).

2(a) Let \(X\) be Ann’s final score, then \(X \sim \text{Bin}(4, 1/2)\) and since \(1 - p = 1/2\) we have the pmf

\[
P(X = k) = \binom{4}{k} \left(\frac{1}{2}\right)^4
\]
(b) The largest probability is when \( k = 2 \) and equals \( P(X = 2) = 3/8 \).

(c) The event that somebody wins 3–1 is the event that \( X = 1 \) or \( X = 3 \) and has probability

\[
P(X = 1) + P(X = 3) = \binom{4}{1} \left( \frac{1}{2} \right)^4 + \binom{4}{3} \left( \frac{1}{2} \right)^4 = 1/2
\]

so it is more likely that somebody wins 3–1. This is similar to the problem of suit distributions in bridge. Here, we are specifying the final result but not who wins; there, we specified the suit distribution but not the order of the suits.

3(a) We have \( P(X > 0) = 1, P(X > 1) = 5/6, P(X > 2) = 4/6, \ldots, P(X > 5) = 1/6 \) and \( P(X > k) = 0 \) for \( k = 6, 7, 8, \ldots \) and \( 1 + 5/6 + \cdots + 1/6 = 3.5 \).

(b) Let \( I_k \) be the indicator for the event that \( X > k \) for \( k = 0, 1, 2, \ldots \) so that

\[
X = \sum_{k=0}^{\infty} I_k
\]

and take expected values to get

\[
E[X] = \sum_{k=0}^{\infty} E[I_k] = \sum_{k=0}^{\infty} P(X > k)
\]