Probabilistic Models, Solutions 2

Practice problems:

1. Introduce the events E: error and L: light comes on. By LTP

$$P(L) = P(L|E)P(E) + P(L|E')P(E') = 0.75 \cdot 0.05 + 0.10 \cdot 0.9 \approx 0.13$$

2(a) The range is $1, 2, 3, ...$ and the probability to get the ace of spades in any given draw is $1/52$. To get $X = k$ we must start with $k - 1$ failures and we get

$$f(k) = \left(\frac{51}{52}\right)^{k-1} \frac{1}{52}, \; k = 1, 2, ...$$

(a geometric distribution with success probability $p = 1/52$).

(b) The range is $1, 2, ..., 52$. The probability that the ace of spades appears in the first trial is $1/52$, hence $f(1) = 1/52$. That it appears in the second trial means that we must draw something else in the first trial, which has probability $51/52$, and then draw the ace of spades from the remaining 51 cards which has probability $1/51$. Hence $f(2) = 51/52 \cdot 1/51 = 1/52$ (formally we have used the formula $P(A \cap B) = P(B|A)P(A)$ here). Continuing in this way we get

$$f(k) = \frac{1}{52}, \; k = 1, 2, ..., 52$$

that is, a uniform distribution. This is intuitively clear because the ace of spades is equally likely to be in any of the 52 positions. Do not confuse conditional and unconditional probabilities here: The (unconditional) probability that the ace of spades appears in the 51st trial is $1/52$ but the *conditional* probability that it appears in the 51st trial *given that it has not appeared before* is $1/2$.

3. Let x denote "not 6" and let X denote your gain. The possible outcome of X and the corresponding sequences of 3 dice are

$X = -1$: xxx, probability $f(0) = (5/6)^3 = 125/216$
$X = 1$, $6xx, x6x, xx6$, probability $f(1) = 3 \cdot (5/6)^2 \cdot (1/6) = 75/216$

$X = 2$, $66x, 6x6, x66$, probability $f(2) = 3 \cdot (5/6) \cdot (1/6)^2 = 15/216$
$X = 3$, 666, probability $f(3) = (1/6)^3 = 1/216$

which gives expected gain

$$E[X] = (-1)\frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = \frac{-17}{216} \approx -0.079$$

Note that X has an "almost binomial" distribution with -1 instead of 0.

4(a) Binomial with $n = 10$ and $p = 0.8$.
(b) Not binomial as trials are not independent (one rainy day makes another more likely)
(c) Not binomial as p changes between months.
(d) Binomial with $n = 10$ and $p = 0.2$.

Turn-in problems

1(a) Call the payout a. We get the equation

$$-\frac{2}{38} = (-1)\frac{36}{38} + a\frac{2}{38}$$

which gives $2a - 36 = -2$ which gives $a = 17$.

(b) $E[X] = (-1)\frac{33}{38} + 6 \cdot \frac{5}{38} = -\frac{3}{38} \approx -0.08$

(c) Denote the payout by a and solve the equation

$$-\frac{2}{38} = (-1)\frac{33}{38} + a \cdot \frac{5}{38}$$

to get $a = 31/5 = 6.2$.

2(a) Let X be Ann's final score, then $X \sim \text{Bin}(4, 1/2)$ and since $1 - p = 1/2$ we have the pmf

$$P(X = k) = \binom{4}{k}(\frac{1}{2})^4$$

for $k = 0, 1, ..., 4$.

(b) The largest probability is when $k = 2$ and equals $P(X = 2) = 3/8$.
(c) The event that somebody wins 3–1 is the event that $X = 1$ or $X = 3$ and has probability

$$P(X = 1) + P(X = 3) = \binom{4}{1}(\frac{1}{2})^4 + \binom{4}{3}(\frac{1}{2})^4 = 1/2$$

so it is more likely that somebody wins 3–1. This is similar to the problem of suit distributions in bridge. Here, we are specifying the final result but not who wins; there, we specified the suit distribution but not the order of the suits.

3(a) We have $P(X > 0) = 1, P(X > 1) = 5/6, P(X > 2) = 4/6, ..., P(X > 5) = 1/6$ and $P(X > k) = 0$ for $k = 6, 7, 8, ...$ and $1 + 5/6 + \cdots + 1/6 = 3.5$.

(b) Let I_k be the indicator for the event that $X > k$ for $k = 0, 1, 2, ...$ so that

$$X = \sum_{k=0}^{\infty} I_k$$

and take expected values to get

$$E[X] = \sum_{k=0}^{\infty} E[I_k] = \sum_{k=0}^{\infty} P(X > k)$$