Probability Models, Test 2, due April 5

1(a) Consider the Wright–Fisher model with $N = 2$ (that is, $2N = 4$) with 3 copies of A and one copy of a in generation 0. Find the probability that A becomes fixed in the first generation and in the second generation, respectively.

(b) Find the probability that A is fixed in the second generation.

(c) Find the probability that fixation of one allele occurs in the first generation.

(d) Now assume variable population size with initial size $N_0 = 2$ as in **(a)**. Do part **(c)** in the cases where population size in the first generation halves and doubles, respectively.

2. Let x and y be any positive numbers, let H be their harmonic mean and M their arithmetic mean.

(a) Show that $H = M$ if $x = y$.

(b) Give three examples where $H < M$.

(c) Use the inequality

$$(x - y)^2 \geq 0$$

to show that $H \leq M$ with equality if and only if $x = y$.

3. Recall the formula

$$E[T] \approx \frac{4N}{\sigma^2} \quad \text{(generations)}$$

from Homework 4, where σ^2 is the limit of the variance in the offspring distribution as $N \to \infty$.

(a) The formula is valid only if the limit of $\mathrm{Var}[X_k]$ exists. For a case where the formula does not apply, suppose one individual is chosen randomly and given $2N$ offspring; all other individuals are given 0 offspring. Show that the limit of $\mathrm{Var}[X_k]$ does not exist (it equals ∞).

(b) In the situation described in **(a)**, what can you say about the actual time T?

(c) Now suppose we choose N individuals randomly and give them 2 offspring each. The remaining individuals get no offspring. Find σ^2 and $E[T]$. Note: For a given individual, you need to find the probability that it is included in the sample of N individuals.

(d) To generalize **(c)**, suppose we choose the fraction f of individuals ($0 < f < 1$) and give them equal numbers of offspring. The remaining individuals get no offspring (thus, in **(c)** we have $f = 1/2$). Show that

$$E[T] = \frac{4Nf}{1-f}$$

(e) Explain intuitively why $E[T]$ increases in **(d)** as f increases (you can do this even if you did not manage to solve **(d)**).